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IMPEDANCE CONTROL OF INDUSTRIAL ROBOTS†

NEVILLE HOGAN

Department of Mechanical Engineering and Laboratory for Manufacturing and Productivity, Massachusetts Institute of Technology, Cambridge, MA 02139, U.S.A.

Manipulation fundamentally requires the manipulator to be mechanically coupled to the object being manipulated. A consideration of the physical constraints imposed by dynamic interaction shows that control of a vector quantity such as position or force is inadequate and that control of the manipulator impedance is necessary. Techniques for planning and control of manipulator behavior are presented which result in a unified approach to target acquisition, obstable avoidance, kinematically constrained motion, and dynamic interaction. A feedback control algorithm for implementing a cartesian end-point impedance on a nonlinear manipulator is presented. The modulation of end-point impedance independent of feedback is also considered. A method for choosing the impedance appropriate to a task using optimization theory is discussed.

NOTATION

Cartesian coordinates

$\mathbf{X}, \mathbf{X}_o, \mathbf{X}_{tol}$	position, commanded position, position tolerance
$\mathbf{V}, \mathbf{V}_o$	velocity, commanded velocity
$\mathbf{F}, \mathbf{F}_{int}, \mathbf{F}_{ext}, \mathbf{F}_{tol}$	force, interface force, external force, force tolerance
M, M_e	inertia tensor, workpiece inertia tensor
$M(\ )$	configuration-dependent inertia tensor
$W(\ )$	configuration-dependent mobility tensor
$\mathbf{p}$	generalized momentum
$S(\ )$	position dependent forces
$V(\ )$	velocity dependent forces
$E(\ )$	elastic force field (curl-free)
$D(\ )$	dissipative force field (curl-free)
m	mass
I	inertia
b, b_{opt}	viscosity, optimum viscosity
k, k_{opt}	stiffness, optimum stiffness
Q	objective function
$\{c\}$	set of control commands
E_k	kinetic energy
E_k^*	kinetic coenergy
$H(\ )$	Hamiltonian
S	strength of zero-mean, Gaussian, purely random process
W	work
Y	generalized admittance
Z	generalized impedance
ρ^2	weighting coefficient

Angular coordinates

$\boldsymbol{\theta}$	absolute angle
$\boldsymbol{\omega}$	angular velocity
$\mathbf{T}, \mathbf{T}_{act}, \mathbf{T}_{int}$	torque, actuator torque, interface torque
$I(\ )$	configuration dependent inertia tensor
$Y(\ )$	configuration dependent mobility tensor
$\mathbf{h}$	generalized momentum
$C(\ )$	inertial coupling torques (coriolis, centrifugal)
$L(\ )$	position transformation equations (linkage kinematics)
$J(\ )$	velocity transformation matrix (Jacobian)
$G(\ )$	accelerative coupling terms

† This paper brings together material which has been presented in part elsewhere.[4, 17–19, 21, 22]

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INTRODUCTION

The use of robots for computer-integrated manufacturing has focused attention on the problems of mechanical manipulation. By any reasonable definition, manipulation requires mechanical interaction with the object being manipulated, and movement tasks may usefully be classified by the magnitude of the mechanical interaction. In some cases the interaction forces are negligible, the instantaneous mechanical work done by the manipulator is negligible, ($dW = \mathbf{F}\, d\mathbf{X} = 0$) and for the purposes of control, the manipulator may be treated as an isolated system. The majority of successful applications of industrial manipulators to date have been restricted to this case; examples are spray-painting and welding[44]. In other cases the interaction forces are not negligible. Recent work in robotics has addressed the problem of manipulation in the presence of a kinematic constraint such as may be encountered during assembly. One solution[31,36] is to control position along the tangent to the constraining surface and to control force along the normal. However, the nature of a pure kinematic constraint is such that along the tangent the interaction forces are zero ($\mathbf{F} = 0$) whereas along the normal into the surface the displacements are zero ($d\mathbf{X} = 0$) and again, as in the case of free movements, the dynamic interaction is zero ($dW = \mathbf{F}\, d\mathbf{X} = 0$). The most general case is that in which the dynamic interaction is neither zero nor neglible ($dW \neq 0$). Almost all manufacturing operations fall into this category: examples include drilling, reaming, routing, counterboring, grinding, bending, chipping, fettling — any task requiring work to be done on the environment. The goal of the research reported here is to understand and control movement and manipulation in the presence of significant dynamic interaction.

IMPEDANCE CONTROL

In the following it will be assumed that the complete controlled system is hierarchically organized: a high-level supervisory system plans movement tasks and presents commands to a lower-level (real-time) controller which operates directly on the manipulator hardware. Seen from the perspective of the high-level supervisor the control is effectively open-loop. The high-level supervisor, while it may have access to sensory data, does not use that data in an immediate feedback control mode to modulate its commands to the lower-level controller during an ongoing task. This arrangement is shown in Fig. 1.

This organization has been proposed as a general form of control and communication for man/machine systems;[41] it is commonly used in industrial robots;[36] and there is evidence that biological motor control systems are similarly organized.[12]

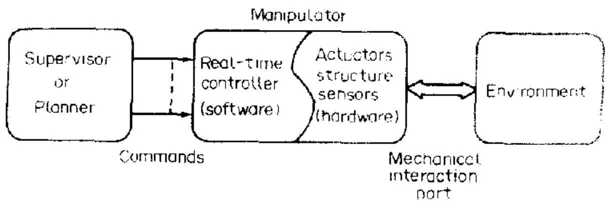

Fig. 1. Schematic diagram of the assumed hierarchical controller structure.

It will further be assumed (without loss of generality) that the controlled manipulator interacts mechanically with its environment across a finite number of interaction ports. An interaction port is defined by a pair of numbers (e.g., force and displacement) which fully characterize both the energetic and the workless interaction between the two systems along a single degree of freedom. An example is a manipulator's end-effector: a vector of six forces and six displacements fully characterize all possible energetic transactions across the end-effector. Strictly speaking, there are six ports associated with the end-effector, one for each degree of freedom; for convenience the entire set will be termed a port. The basic philosophy of impedance control is to describe the behavior of the physically interacting systems and the design of the controller in terms of the variables of the interaction port.

The manipulator is some collection of physical structures, actuators and sensors (hardware) combined with some set of feedback processors, compensators and control algorithms (software). A unified framework for considering the action of both hardware and software is obtained by noting that while the controller software may moify the apparent behavior of the physical hardware, the controlled manipulator is still a physical system and must obey the constraints of the real world. As the manipulator hardware must obey the constraints of the real world the manipulator controller should not attempt to violate them. Of all the functions or algorithms or differential equations which could be implemented in the software, only those which yield a behavior seen from the interaction port which satisfies the constraints imposed on real physical hardware need be considered.

Along each degree of freedom, power flow between an element (or system) and its environment is always defined by the product of two conjugate variables, an effort (e.g., a force, a voltage) and a flow (e.g., a velocity, a current).† The first important physical

† The notation and formalism of Paynter's Bond Graphs[24,37,39] are used throughout this paper.

constraint is that no one system (or element) may determine both variables of a single interaction port; a manipulator may impress a force on its environment or impose a displcement or velocity on it, but not both.

The second important physical constraint arises from the energetic nature of the dynamic element or system connected to the interaction port, which may place restrictions on which variables may be defined by which system. For example, if one of the interacting systems is an inertia, its physical behavior is such that it produces motions in response to applied forces and not vice versa. The defining property of a (generalized) inertia is storage of kinetic energy, the integral of (generalized) velocity with respect to (generalized) momentum (see Appendix I). The fundamental constitutive equation of a kinetic storage element is:

$$\mathbf{V} = Y(\mathbf{p}). \tag{1}$$

The only restriction on this relation is that it be integrable to define kinetic energy; it need not be invertible. As the rate of change of momentum is a function of applied force, this element is fundamentally an admittance, a system described by a relation, possibly nonlinear and/or dynamic, which accepts effort (e.g., force, voltage) inputs and yields flow (e.g., motion, current) outputs. The dual or conjugate of an admittance is an impedance, a relation between flow (motion) inputs and effort (force) outputs. The concepts of admittance and impedance are commonly used in linear electric circuit theory, but in the linear case the distinction between admittance and impedance is obscured as one is simply the inverse of the other. In the more general nonlinear case the inverse of a functional relation need not exist and the distinction between admittance and impedance is essential.

For example, a kinematically constrained linkage is an inertial system capable of storing kinetic which may not be describable as an impedance. Consider an interaction port at the tip of a planar linkage shown in Fig. 2(a).

The kinematic equations are a transformation from joint variables $\{\theta1, \theta2\}$ to port variables $\{X1, X2\}$:

$$\mathbf{X} = L(\boldsymbol{\theta}). \tag{2}$$

This transformation may not be invertible (e.g., if $\theta1 - \theta2 = n\pi$, n any integer) and for a given linkage there exists a two-dimensional infinity of points in $\{X1, X2\}$ for which no point in $\{\theta1, \theta2\}$ exists. As a result, it may not be possible to impose an arbitrary displacement (or velocity) on the interaction port. On the other hand it is always possible to push on it, and the applied force results in a torque about the joints given by:

$$\mathbf{T} = J^t(\boldsymbol{\theta})\,\mathbf{F}. \tag{3}$$

This torque results in a motion of the linkage which in turn results in a motion of the end-point. The *causal constraint*[21] becomes even more binding if the linkage has more degrees of freedom than the end-point. Consider the planar linkage shown in Fig. 2(b). In this case,

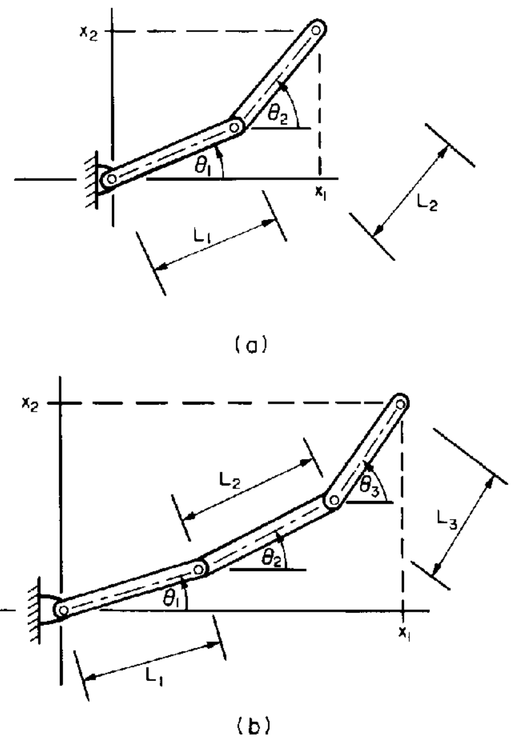

Fig. 2. (a) A planar two-member linkage, and (b) a planar three-member linkage.

the inverse transformation *never* exists. It may be possible to displace the interaction port but this information alone is insufficient to determine the "internal" displacements of the mechanism. In contrast, the corresponding transformation from forces applied at the interaction port to the consequential torques applied to the links is always well defined, and the dynamic equations can be solved to determine the resulting motion of the end point (see Appendix I). This sytem is an admittance.

As the controlled manipulator (hardware plus software) is subject to physical constraints it must be either an impedance (e.g., spring-like behavior — motion in, force out) or an admittance (e.g., mass-like behavior — force in, motion out). Furthermore, if the environment is an admittance the manipulator must be an impedance (or vice versa).

What this means in practical terms is that the controlled manipulator must assume a behavior which is the complement of that of the environment. In virtually all practical manipulation problems the behavior of the environment as seen by the manipulator is that of an admittance, i.e., a mass or a kinematic constraint will be present. Consequently, to ensure compatibility with its environment the behavior of the manipulator as seen through the interaction port should be that of an impedance. Although there are many real, practical situations in which the environment is such that to a good approximation a manipulator may impose displacements on it (e.g., move a small unconstrained mass around), a significant number of practical situations exist (e.g., constrained manipulations) in which this manipulator behavior would be inappropriate or, worse yet, damaging.

An alternative solution might be to change the structure and/or parameters of the controller as the conditions imposed by the environment change. This would require the controller to monitor the environment continuously, identify changes, and adapt its own behavior accordingly — a far-from-trivial task. Changes in the structure and parameters of the environment may take place very rapidly (consider the transition from free motion to constrained motion as an object comes in contact with a surface) and there may not be sufficient time for the lengthy process of system identification. On the other hand, if the controller is structured so that it always impresses a force on the environment in relation to its motion (that is, it behaves as an impedance) there are no practical situations in which its behavior is inappropriate, no practical task has been excluded, and the need to identify the structure of the environment has been reduced.

It is even more important to note that the assumption that the environment is an admittance, e.g. an inertia, is sufficient to permit the decomposition of the dynamic behavior of the manipulator seen at the interaction port into components which may be reassembled by linear addition *even when the behavior of any or all of the components is nonlinear*. The net force acting on an inertia is a sum (possibly signed) of all of the individual forces acting on it. The net impedance coupled to an inertia is *always* a simple sum (unsigned) of the individual impedances coupled to it.

Because the mechanical interaction with the environment will change with different tasks, or even in the course of a single task — the manipulator may be coupled to the environment in one phase and decoupled from it in another — the behavior of the manipulator should be adaptable. The controller should be capable of modulating the impedance of the manipulator as appropriate for a particular phase of a task. Furthermore, just as a position-controller is designed to maintain the actual output position close to some desired position in the face of disturbances and/or parameter changes, the manipulator controller should be designed to maintain the actual interaction-port impedance close to a desired impedance in the face of disturbing influences. What is required in general is the modulation and control of impedance;[16–23] hence the title of this paper.

CARTESIAN IMPEDANCE PLANNING

In conventional manipulator control a high-level supervisor plans motions (possibly off-line) and dictates the desired values of a vector of interaction port variables such as position, velocity or force which a low-level (real-time) controller then implements. The supervisor of an impedance-controlled manipulator must likewise dictate nominal values for a vector of port variables, but must also go a step further to plan the manipulator impedance, the dynamic equation describing its interactive behavior.

Cartesian path planning is an effective strategy for controlling free (non-interactive) motions of a manipulator. Simply stated, the procedure is to plan the desired motions of the manipulator in terms of the motion of a point on the end-effector in cartesian workspace coordinates. The planned trajectory is then translated into actuator coordinates for execution. Algorithms have been developed to deal with the fact that the end-effector is not a geometric point but has extent and shape and must be steered around obstacles and threaded through tight spaces.[28] A substantial body of literature has been published on methods for implementing a planned cartesian path.[35, 36] The approach is widely used in the control of industrial manipulators and there is evidence of a comparable strategy of motion control in biological systems.[1, 32]

Mechanical interaction may also be planned and controlled in cartesian end-point coordinates. The planner assumes that the end-effector impedance expressed in cartesian workspace coordinates has a conveniently simple form. To ensure dynamic feasibility the choice of the simplified form to be assumed during planning is based on the dominant dynamic behavior of the manipulator. The choice is a tradeoff between keeping the complexity of the planning within manageable limits while ensuring that planned behavior adequately reflects the real dynamic behavior of the controlled system. As a result it depends both on the manipulator itself and on the environment in which it operates. For example, a manipulator intended for underwater applications will operate in a predominantly viscous environment and it may be reasonable to ignore inertial effects during the planning phase. In contrast, a manipulator intended for operation in a free-fall orbit will encounter a predominantly inertial environment. For terrestrial manufacturing applications both gravitational and inertial effects are important, and the dominant dynamic behavior is second order in displacement along each degree of freedom.

As the environment is an admittance the manipulator should be an impedance and the simplest form of the behavior of the coupled system (manipulator and environment) is that of a mass driven by motion-dependent forces. When the manipulator is decoupled from its environment the terms due to the environmental admittance disappear, and in principle the manipulator alone need exhibit no mass-like behavior. In practice the uncoupled manipulator still has inertia (albeit nonlinear and configuration-dependent). Thus, in reality, the structure of the coupled and uncoupled dynamic equations are similar — that of an admittance coupled to an impedance. In effect, the unavoidable inertial effects of the machine structures place a lower limit on the admittance seen by the controllable impedance of the manipulator — it is never zero. There is always an admittance which sums both forces and impedances. However, though the inertial effects may not be eliminated, the apparent inertial behavior of the end-point may be modified.

Most manipulatory tasks are fundamentally described in relative coordinates, that is, in terms of displacements and rotations with respect to a workpiece, tool or fixture whose location in the workspace is not known in advance with certainty. As a result, planning is simplified if the end-point inertial behavior is modified to be that of a rigid body with an inertia tensor which remains invariant under translation and rotation of the coordinate axes. This is achieved if:

$$M = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \quad (4)$$

This is the inertia tensor of a rigid body such as a cube of uniform density. This inertia tensor eliminates the angular velocity product terms in the Euler equations for the motion of a rigid body[8] and ensures that if the system is at rest the applied force and the resulting acceleration vectors are co-linear.

To represent the dominant second-order behavior of the system the interface forces due to the manipulator impedance depend only on displacement, velocity and time:

$$\mathbf{F}_{int} = F(\mathbf{X}, \mathbf{V}) - M\, d\mathbf{V}/dt \quad (5)$$

All of the parameters in this expression are assumed to be functions of the set of control commands $\{c\}$.

If the planner is concerned only with the nodic component[21] of the manipulator impedance (that which can be moved around the workspace) the equations of motion may be written in terms of a displacement from a commanded position $\mathbf{X}_o$:

$$\mathbf{F}_{int} = F(\mathbf{X}_o - \mathbf{X}, \mathbf{V}_o - \mathbf{V}) - M\, d\mathbf{V}/dt. \quad (6)$$

Although there are cases in which coupled nonlinear viscoelastic behavior is useful, often the position- and velocity-dependent terms may be separated:

$$\mathbf{F}_{int} = S(\mathbf{X}_o - \mathbf{X}) + V(\mathbf{V}_o - \mathbf{V}) - m\, d\mathbf{V}/dt. \quad (7)$$

A further simplification is to assume that the position-dependent terms are curl-free.† A potential function is then definable which may be interpreted as stored elastic energy. A similar set of assumptions permits the velocity-dependent terms to be described as a dissipative potential field. The dynamics of the end-effector have now been reduced to:

$$M\, d\mathbf{V}/dt = E(\mathbf{X}_o - \mathbf{X}) + D(\mathbf{V}_o - \mathbf{V}) - \mathbf{F}_{int}. \quad (8)$$

If the environment is a simple rigid body acted on by unpredictable (or merely unpredicted) forces, the dynamic equations for the environment are:

$$M_e\, d\mathbf{V}/dt = \mathbf{F}_{ext} + \mathbf{F}_{int} \quad (9)$$

and the coupled equations of motion are:

$$(M_e + M)\, dV/dt = E(X_o - X) + D(V_o - V) + F_{ext}. \quad (10)$$

Because of the simple form of the dynamic equations the elastic field and the external forces are sufficient to define static stability. Asada[5] has shown how elastic potential fields may be used as the basis of an approach to planning stable grasp. Stable equilibrium configurations of end-effector and workpiece are defined by finding minima of the potential energy function. Gravitational forces are readily included by expressing them as a potential function and combining it with the potential function of the manipulator by simple addition. Note, however, that the dynamic stability of the end-effector is not guaranteed (that is, in principle, sustained oscillations are possible). To ensure dynamic stability the dissipative field must be chosen appropriately, that is, the complete impedance must be planned, not just the elastic behavior.

†For each component of $S(\ \ )$ and each component of $\mathbf{X}$, the crossed partial derivatives are identical.

VIRTUAL INTERACTION

Because the planned impedance is simple the (nominal) behavior of the manipulator is easily predicted and impedance control can be used to advantage even when the systems do not physically interact. "Virtual" (or non-contact) interaction can be planned in terms of an impedance specifying the accelerating forces to be applied to the total mechanical admittance as a function of the position and velocity of the end-effector. Successive target locations (i.e., a path) may be specified by means of a (time-varying) potential field with stable equilibria at the target locations. Each single command specifies a potential function which is a "valley" with its bottom at the target. This "valley' is depicted by a map of iso potential contours in Fig. 3(a).

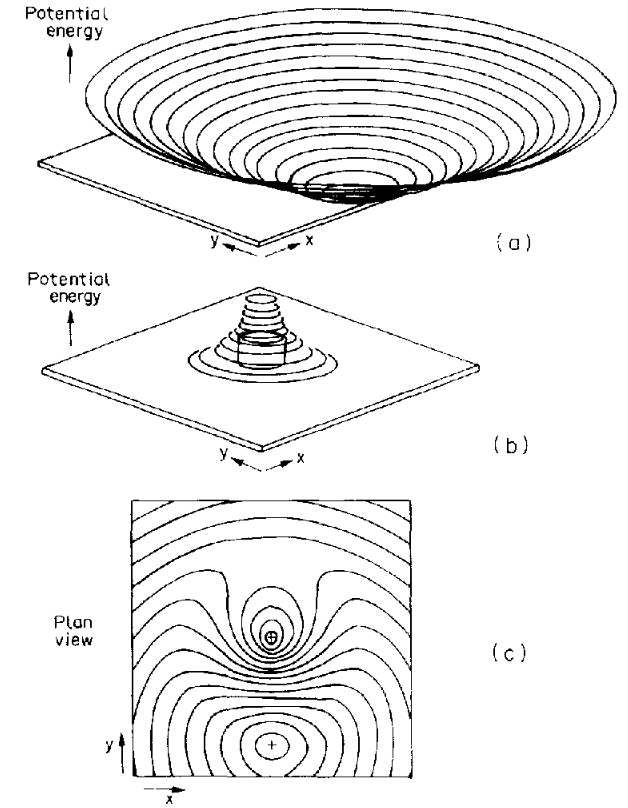

Fig. 3. Isopotential contour maps of the static component of commanded impedances which may be used for: (a) target acquisition, (b) obstacle avoidance, and (c) simultaneous target acquisition and obstacle avoidance.

Conversely, an obstacle or a region in the workspace to be avoided may be specified by means of a potential field with an unstable equilibrium point at the obstacle. The command set specifies a potential function which is a "hill" centered over the obstacle (see Fig. 3b).

The target-acquisition command and the obstacle-avoidance command could be combined in a number of ways, but remember that the admittance sums the impedances. The inevitable inertial behavior of the end-point guarantees the superposition of the components of the impedance-controller action *independent of the linearity of the components*. It is always possible to plan obstacle-avoidance and target-acquisition (or any other aspect of the complete task) independently and then combine all commands at execution time by simply adding the impedances. in this case the corresponding potential fields (see Fig. 3c).[18,19] Of course, neither targets nor obstacles need stay fixed in the workspace and this approach may be used to make a manipulator avoid "invaders", objects which may move about the work-space in an unpredictable (or merely unpredicted) manner.[3,4]

To be of practical value, the repulsive force fields used to control virtual interaction must be nonlinear; the repulsive force must drop to zero for sufficiently large separations between the end-effector and objects in the environment (see Fig. 3b). Note that this is precisely the type of non-invertible, nonlinear force/displacement behavior for which no inverse (compliance) form exists. The concept of giving the manipulator end-point the behavior of a generalized compliance has been discussed in the literature under the general heading of "compliance", "compliant motion control" or "force control".[31,36] The control strategy presented in this paper is considerably more general. If the end-point dynamic behavior is expressed as an impedance, the above obstacle-avoidance behavior is included as a special case. If it were expressed as a compliance (or more generally, an admittance) this useful behavior would be excluded.

The use of potential functions as commands to a robot is similar to the approach used by Khatib[25] to navigate a manipulator through a complicated environment. The distinguishing feature (and advantage) of impedance control is that the same controller used to deal with free motions can also be used to deal with real mechanical interaction. The success of impedance control as a unifying framework for dealing with both real interaction (kinematically constrained manipulation) and virtual interaction (including avoiding moving "invaders") has been demonstrated by performing both of these tasks in real time using a spherical coordinate manipulator.[illegible] The same controller was used for both tasks and the algorithm was simple enough to be implemented using 8-bit 2 MHz microprocessors (Z-80, one for each axis) for the real-time controller. One example of the obstacle-avoidance behavior achieved is shown in Fig. 4.

Because of the imposed simple form of the dynamic equations the visualization, prediction and planning of the system behavior has been radically simplified. For example, in the absence of external active sources the total energy of the system, kinetic plus potential, may never increase. This permits easy prediction of the maximum velocities which may result from a given set of commands without computing the detailed trajectories. Conversely, as the potential energy function is one of the commands, it is readily chosen so that a desired maximum velocity is never exceeded. If the command is given when the system is at zero velocity (e.g., a workpiece has just been grasped) then it is not even necessary to know the mass of the grasped object.

The impedance control strategy described above may be a key element in the integration of Computer-Aided Design with Automated Manufacturing. Design is more than mere part description and requires consideration of aspects of part manufacture such as ease of assembly. One way for a machine designer to understand the constraints imposed by an automated manufacturing operation such as robotic assembly is to plan the commands necessary for a robot to assemble machine components during the process of designing them. If vital physical insight is to be gained the designer must not be distracted by the true nonlinear dynamic complexity of the robot. Cartesian impedance control masks the true dynamics of the robot so that its apparent behavior is simple and easy to predict — that of a mass moving under the action of springs and dampers in cartesian coordinates. In addition, as most machine designers are visual thinkers, graphic display of the robot dynamic behavior is essential. Visualization of planned behavior via potential functions promises to be a powerful technique for graphic display to a machine designer of the consequences of a given set of commands to a robot.[19] This approach to the integration of CAD and CAM is being explored.[38]

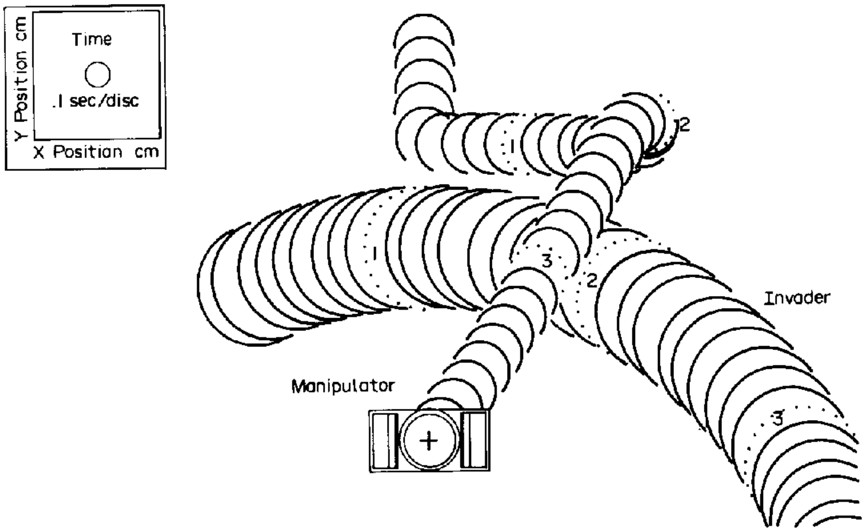

Fig. 4. Avoidance of an unpredictably moving "invader" by a spherical-coordinate manipulator controlled by 8-bit, 2 MHz microprocessors. Successive positions of the manipulator end-effector and the invader in the vertical plane at 100 msec intervals are shown. All of the behavior shown here is the robot's response to a single impedance command from the supervisor.

CARTESIAN IMPEDANCE CONTROL

Given a planned cartesian end-point impedance, how may it be implemented? One approach, presented below, is to express the desired cartesian coordinate impedance in actuator coordinates and then use a model of the manipulator dynamics to derive the required controller equations. Assume the desired behavior is that of eq. (8). The kinematic relation between actuator (e.g., joint) coordinates and cartesian end-point coordinates provides sufficient information to express the desired behavior in actuator coordinates. Then, assuming that the kinematic, inertial, gravitational and frictional effects provide an adequate model of the manipulator dynamics as follows:

$$I(\boldsymbol{\theta})\, d\boldsymbol{\omega}/dt + C(\boldsymbol{\theta},\boldsymbol{\omega}) + V(\boldsymbol{\omega}) + S(\boldsymbol{\theta}) = \mathbf{T}_{act} + \mathbf{T}_{int}, \tag{11}$$

an expression for the required actuator torque as a function of actuator position and velocity and end-point force can be derived (see Appendix II):

$$\begin{aligned}\mathbf{T}_{act} = {} & I(\theta)\, J^{-1}(\theta)\, M^{-1}\, E[\mathbf{X}_0 - L(\theta)] + S(\theta) \\ & + I(\theta)\, J^{-1}(\theta)\, M^{-1}\, D[\mathbf{V}_0 - J(\theta)\,\omega] + V(\omega) \\ & + I(\theta)\, J^{-1}(\theta)\, M^{-1}\, \mathbf{F}_{int} - J^{t}(\theta)\, \mathbf{F}_{int} \\ & - I(\theta)\, J^{-1}(\theta)\; G(\theta,\omega) + C(\theta,\omega). \end{aligned} \tag{12}$$

This equation expresses the desired behavior as a nonlinear impedance in actuator coordinates, independent of how it is to be implemented. For the purpose of designing a control algorithm it may be viewed as a nonlinear feedback law relating actuator torques to observations of actuator position, velocity and interface force. The input (command) variables are the desired cartesian position (and velocity) and the terms of the desired (possibly nonlinear) cartesian dynamic behavior characterized by M, $D(\ )$ and $E(\ )$.

The feasibility of cartesian impedance control has been investigated[7, 22] by implementing this nonlinear control law to impose cartesian end-point dynamics on a servo-controlled, planar, two-link mechanism (similar to the nonlinear linkage in a SCARA† robot). A comparison of computer simulations with hardware performance showed that the algorithm remains stable in the face of substantial errors in the parameters of the manipulator model, although there is some degradation in performance.[7] A simple analysis estimating the computation required to implement this controller on a six-degree-of-freedom manipulator indicated that the computational burden is comparable to "exact" approaches to generating manipulator commands such as the recursive LaGrangian[14] and Newton–Euler[35] methods or the configuration space method.[13]

Inverting the kinematic equations of a manipulator to determine a desired time-history of positions in actuator (joint) coordinates has been described[36] as one of the most difficult problems in robot control. For some manipulators (e.g., those with non-intersecting wrist joint axes) no closed-form solution may be possible. A key point to note is that the above algorithm eliminates the need to solve the inverse kinematic problem. The desired end position is a known command from the supervisor; actual end-point position is determined from actuator (joint) position; the required actuator force (torque) is determined from the desired end-point force, which is in turn determined from the desired end-point impedance. This is a direct consequence of the care which was taken to ensure that the desired behavior was compatible with the fundamental physics of manipulation and was expressed as an impedance.

If the interface forces and torques in eqs (8) and (12) are eliminated and the position- and velocity-dependent terms reduced to linear diagonal forms, this implementation of impedance control bears some resemblance to the resolved acceleration method.[29] However, unlike the resolved acceleration method, the impedance control algorithm presented above is based on desired end-point behavior which may be chosen rationally using approaches such as the optimization technique presented below. Furthermore, the impedance control algorithm includes terms for coping with external "disturbances". Without the external "disturbance" terms (which have no counterpart in the resolved acceleration algorithm) the manipulator is not capable of controlled mechanical interaction with its environment. Note also that the above approach to defining the controller equations is not restricted to commanded linear behavior and can be applied equally well to achieve the more general coupled nonlinear behavior of eq. (5).

It is not claimed that the above algorithm is the only or the best way to achieve a desired end-point impedance. It is presented here only to demonstrate that a control law capable of modulating the end-point impedance of a manipulator may be formulated. Other approaches to controller design such as the model-referenced adaptive control method[11] may be used. For impedance control, the reference model is the desired impedance. It is expressed in end-point coordinates rather than in joint coordinates[11] but this should present no fundamental obstacle to its implementation.

It is important to remember that feedback control strategies represent only one possible way of modulating manipulator dynamics. An alternative is to use the intrinsic properties of the end-effector[9, 10, 49] or the manipulator hardware. For example, kinematic redundancies may be used to provide a measure of control over the inertial component of the end-point dynamics. As always, it is important to remember that an inertia is fundamentally an admittance; flow (velocity) is determined as a response to impressed effort (force) and dealing with kinematic redundancy is considerably simplified if the inertia is written in its fundamental causal form as a relation determining generalized velocity as a function of generalized momentum rather than vice-versa:

$$\omega = Y(\theta)\, \mathbf{h} \tag{13}$$

where

$$Y(\theta) = I^{-1}(\theta). \tag{14}$$

$Y(\theta)$ is the linkage inertia written in admittance form. To help distinguish between the impedance and admittance

† Selective Compliance Assembly Robot Arm.[30]

forms of the inertia tensor, the term "mobility" will be used for the admittance form.

At a fixed configuration, the kinematic transformations between joint angles and end-point cartesian coordinates define not only the relations between generalized displacements, flows and efforts in the two coordinate frames, they also define the relations between generalized momenta:

$$\mathbf{h} = J^t(\boldsymbol{\theta})\,\mathbf{p}. \tag{15}$$

These relations may be used to write the mobility in end-point coordinates as follows:

$$\mathbf{V} = W(\boldsymbol{\theta})\,\mathbf{p} \tag{16}$$

$$W(\boldsymbol{\theta}) = J(\boldsymbol{\theta})\,Y(\boldsymbol{\theta})\,J^t(\boldsymbol{\theta}). \tag{17}$$

Note that $J(\boldsymbol{\theta})$ in these equations need not be square. The physical meaning of the end-point mobility tensor is that if the system is at rest (zero velocity) then a force vector applied to the end-point causes an acceleration vector (not necessarily co-linear with the force) which is obtained by premultiplying the force vector by the mobility tensor (see Appendix I).

The control over the end-point inertial dynamics afforded by redundant degrees of freedom can be represented graphically through the ellipsoid corresponding to this tensor. The locus of deviations of the generalized momentum from zero for which the kinetic energy is constant is an ellipsoid, the "ellipsoid of gyration"†.[46] As shown in Appendix I, the eigenvalues and eigenvectors of the symmetric matrix $W(\boldsymbol{\theta})$ define the size, shape and orientation of the ellipsoid of gyration in end-point coordinates (see Fig. 5a).

To illustrate the modulation of the end-point mobility using linkage redundancy, consider the planar three-link mechanism shown in Fig. 2(b). Assuming the links are rods of uniform density with lengths in the ratio of 1 : 2 : 3, Figs. 5 (b–d) show the effect on the ellipsoid of gyration of changes in linkage configuration for a fixed position of the end point.

An alternative representation of inertial behavior is via the ellipsoid of inertia.[46] Asada[6] has suggested its use as a tool for designing robot mechanisms. However, the ellipsoid of gyration is the more fundamental representation; it is readily obtained even when the Jacobian of the linkage is non-invertible. Also, while the matrix $Y(\boldsymbol{\theta})$ may never have zero eigenvalues, (assuming real links with non-zero mass) the matrix $W(\boldsymbol{\theta})$ may, because of the kinematics of the linkage. Thus the end-point inertia tensor, the inverse of the mobility tensor, does not exist for some linkage configurations. If the inertial behavior of the tip is written in the conventional (impedance) form:

$$p = M(\boldsymbol{\theta})\,\mathbf{V} \tag{18}$$

there exist locations in the workspace for which the eigenvalues of the tensor $M(\boldsymbol{\theta})$ become infinite. On the other hand the worst the eigenvalues of $W(\boldsymbol{\theta})$ will do is go to zero, which is easier to deal with computationally. Again, this is a reminder of the fact that the difference between impedance and admittance is fundamental and that an inertia is an admittance, not an impedance.

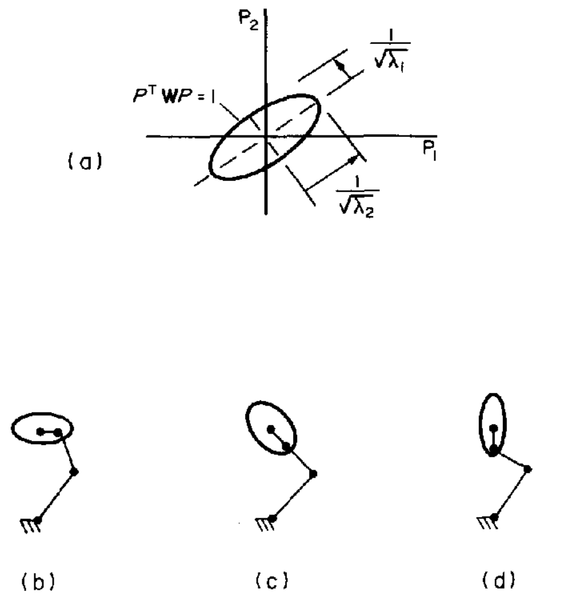

Fig. 5. Schematic representation of the influence of kinematic redundancies on the mobility of the end-point (the inverse of its effective mass). The ellipsoid of gyration associated with the mobility tensor is shown in (a). The eigenvalues are inversely proportional to the effective mass in the direction of the corresponding eigenvectors and the square root of their ratio determines the ratio of the major and minor axes, which are co-linear with the eigenvectors. For a planar, three-member linkage with links of uniform density and cross section and lengths in the ratio 1 : 2 : 3 the effect on the ellipsoid of gyration of changing the linkage configuration for a fixed position of the end-point is shown in (b), (c), and (d).

CHOOSING THE OPTIMUM IMPEDANCE

The impedance appropriate for a given situation depends on the task to be performed. Many manipulatory tasks are characterized by a tradeoff between allowable interface forces and allowable deviations from desired motions. Whether it has been rationally chosen or not, the manipulator impedance specifies how that tradeoff will be made. As a result one way of determining the

† Also known as Binet's ellipsoid.

appropriate impedance is to specify the task as an objective function to be optimized. The appropriate impedance may then be determined using optimization theory. The derivation of optimum impedances for general manipulatory tasks is a topic of some complexity, but to illustrate the technique two simplified examples are presented which provide considerable insight into the mechanics of manipulation.

In the first example consider a single degree-of-freedom, assume that both the machine impedance and the environmental admittance are linear passive dissipative elements. The environmental admittance is:

$$V = YF. \tag{19}$$

The machine impedance is nodic,[21] that is, its output force is a function of deviation from a variable (commanded) reference velocity.

$$F = Z(V_0 - V). \tag{20}$$

Combining equations

$$V = YZV_0(1 + YZ) \tag{21}$$
$$F = ZV_0/(1 + YZ). \tag{22}$$

First assume the task is to maximize the transmission of power into the environment.

Objective: Maximize $P = FV = YZ^2V_0^2/(1 + YZ)^2$ (23)

Maximize P w.r.t. V_0: $V_0 = \max V_0$ (maximize commanded velocity) (24)
Maximize P w.r.t. Z: $Z = \max Z$ (maximize machine impedance) (25)
Maximize P w.r.t. Y: $ZY = 1$. (26)

Maximizing with respect to V_0 and Z only yield conditions for a maximum if the system parameters or variables are bounded, and the first two conditions state essentially that the machine should operate at the limits of its performance. Because in this simple linear static case admittance is the reciprocal of impedance, the third condition states that machine impedance should match environmental impedance (after the first two conditions have been satisfied). This is a familiar result and is a design rule of great versatility, applicable in any situation in which a source is to impart maximum power to a load.

Another common manipulatory task is to make precise movements. However, if robot and workpiece interact mechanically, a precise path controller may generate excessive forces (e.g., if the workpiece is kinematically constrained). Thus a common task objective is to minimize deviations from desired motion while simultaneously minimizing interface force. Assume this objective may be modelled as minimizing a weighted sum of squares of deviations of interface force from zero and velocity from its commanded value.

Objective: Minimize $Q = p^2(V_0 - V)^2 + F^2$. (27)

The weighting coefficient p^2 specifies the allowable tradeoff between interface force and motion error.

$$Q = (p^2 + Z^2)V_0^2/(1 + YZ)^2$$

Minimize Q w.r.t. V_0: $V_0 = \min V_0$ (minimize commanded velocity) (28)
Minimize Q w.r.t. Y: $Y = \max Y$ (maximize environmental admittance) (29)
Minimize Q w.r.t. Z: $Z = p^2Y$. (30)

Minimizing with respect to V_0 and Y only yield conditions for a minimum if the system variables or parameters are bounded, and the first two conditions state that the system should operate on these bounds. The third condition states that machine impedance should be proportional to environmental admittance. This may be considered as a designer's "rule of thumb" for manipulation. With appropriate scaling of force and motion the weighting coefficient may be made equal to unity, and the design rule is: "match machine impedance to environmental admittance". If the environment is unyielding, a manipulator should accomodate the environment; if the environment offers little resistance, the manipulator may impose motion upon it.

Although these results were obtained using an extreme simplification of the mechanics of manipulation, this simple static analysis captures the essence of the interaction between manipulator and environment, and yields an intuitively satisfying result: that power transmission and manipulation (at least insofar as manipulation is modeled by the objective function of eq. 27) are fundamentally conflicting task requirements. In view of the fact that a manipulator must be versatile — it may be called upon to transmit power in one phase of a working cycle (e.g., accelerate or decelerate a workpiece as fast as possible) and manipulate at another (e.g., assemble the workpiece to another) — a controllable mechanical impedance is imperative.

Now consider a simple dynamic case. Assume the environment is a rigid unconstrained workpiece and that the dynamics of the coupled system are given by eq. (10) with linear elastic and viscous terms. If the inertia tensor of the workpiece, M_e, has the form of eq. (4) then the combined inertia tensor, $M_e + M$, also has this form. In this case the coordinate axes of the end-point may be rotated to suit the task without generating off-diagonal terms in the inertia tensor, i.e., without introducing any inertial coupling between degrees of freedom of the end-point.† The inertia tensor of the workpiece will not in general have this form, but because of the enormous simplification afforded by decoupling the degrees of freedom, (e.g., accelerations co-linear with applied forces) it may be worthwhile to tune the apparent end-point inertia of the manipulator so that the combined inertia tensor of the coupled system has the form of eq. (4). In the following example this will be assumed.

If a potential field is definable (as in the target-acquisition and obstacle-avoidance examples above) then the stiffness is symmetric and it is always possible to rotate the coordinate axes so as to diagonalize the stiffness tensor. If in addition the viscosity tensor is symmetric and has eigenvectors co-linear with those of the stiffness tensor, the general six degree-of-freedom problem decomposes into six single degree-of-freedom problems and each degree of freedom may be dealt with separately. In the following example only a single degree of freedom will be considered.

Consider the task of maintaining a fixed position in the face of unpredictable (or merely unpredicted) perturbations from the environment. (These might be due to excitation forces from a power tool.) To reflect the paucity of *a-priori* information about the perturbations they will be modeled as a zero-mean, Gaussian, purely-random process of strength S.

The tradeoff implicit in this task will be described (as before) as minimizing the sum of squares of forces and motion errors. The objective function is:

$$Q = \int_0^\infty \{(F/F_{tol})^2 + [(X_o - X)/X_{tol}]^2\}\, dt. \qquad (31)$$

F_{tol} is an allowable force tolerance, X_{tol} is an allowable position tolerance. Minimization of this objective function with respect to the impedance parameters k and b is presented in Appendix III and yields the following results:

$$k_{opt} = F_{tol}/X_{tol} \qquad (32)$$

$$b_{opt} = \sqrt{(2\, k_{opt}\, m)}. \qquad (33)$$

In this simple case the optimum stiffness is equal to the ratio of force tolerance, F_{tol}, to position tolerance, X_{tol}. With no penalty on velocity errors, the optimum damping is such as to yield a second-order Butterworth pole configuration.

Viewed simply as an optimization problem, these results are the well-known solution to the second-order feedback regulator problem.[26] Their importance in this context is two-fold: First, they demonstrate that a trade-off modeled by an objective function such as eq. (31) can be used to determine the appropriate manipulator impedance. The analysis expresses the required impedance command to the manipulator in terms of readily obtained meaningful quantities — the force and position tolerances — associated with the task.

Second, and more important, the results are expressed in terms of the mechanical behavior of the end-effector *regardless of how that behavior is achieved.* The optimal impedance may be implemented by any means, using a feedback control law or manipulator redundancies or otherwise, permitted by a given manipulator design. For example, the primate neuromuscular system has the capacity to change its mechanical impedance by simultaneous activation of opposing muscles[27,50] and the above analytical technique has been used to derive a prediction of this behavior consistent with experimental observation.[15,20]

In this simple analysis the external forces were almost completely unmodeled as the assumption of a purely random process is tantamount to an assumption of complete unpredictability. The analysis demonstrates that even with extremely little information about the environment, the interaction between manipulator and environment may be controlled so as to meet task specifications. Naturally, the more information about the environment that is available, the better one would expect the system performance to be. However, this suggests the tantalizing possibility that the impedance may be chosen to tradeoff performance against need for information about the environment. This is a topic for further research.

SUMMARY AND DISCUSSION

This paper has presented a fundamental approach to manipulation. Because by its nature manipulation requires mechanical interaction between systems, the focus of the approach is on the characterization and control of interaction. By assuming that no control system may make a physical system behave like anything other than a physical system, several simple but fundamental observations may be made: Command and control of an interaction port vector such as position or force is not enough to control dynamic interaction between the systems; the controller must also command

† Of course the gyroscopic terms accompanying high-speed angular velocity of the end-point have not been eliminated.

and control a relation between port variables; in the most common case in which the environment is an admittance (e.g., a mass, possibly kinematically constrained) that relation should be an impedance, a function, possibly nonlinear, discontinuous and/or dynamic, specifying the force produced in response to a displacement or velocity imposed by the environment. Even more important, if the environment is an admittance, the total impedance is expressible as a sum of component impedances, even when the components are nonlinear.

The techniques outlined in this paper are an extension of conventional planning and control strategies. The essence of the approach to cartesian impedance planning is to assume a controller capable of imposing a simple behavior on the naturally complicated behavior of the end-effector. The choice of a realistic but appropriately simple form for the imposed impedance makes the problems of predicting and planning the behavior of the complete controlled system (manipulator and environment) tractable. Simplifying assumptions of symmetry permit the general end-point behavior to be decomposed along orthogonal degrees of freedom. Restricting attention to impedances with exact differentials (force fields with zero curl) permits the definition of potential functions. In a sense, the use of potential functions maps dynamics into statics. This opens up new (and considerably simpler) ways of planning and performing tasks such as obstacle avoidance.

Because impedance is a relation between the variables of the interaction port it embodies an allowable tradeoff between their values which is implicit in the specification of many manipulatory tasks. As a result optimization techniques may be used to derive the impedance appropriate to a task from a specification of the task. A simple but revealing static example was presented which demonstrated a fundamental difference between power transmission and manipulation and (by analogy) highlighted the need for impedance control in manipulation. In a simple dynamic example, stochastic optimization theory was used to express the appropriate impedance in terms of force and motion tolerances independent of the way the impedance is implemented, e.g., without assuming feedback control. The method is general and has been applied to a nonlinear system.[15, 20]

Once selected, an impedance may be implemented in a number of ways, using to advantage the resources of a specific manipulator. A feedback control algorithm for imposing cartesian impedance on a general nonlinear manipulator was presented. Because care was taken to express the desired behaviour as an impedance, compatible with the fundamental mechanics of manipulation, solving the inverse kinematics problem is unnecessary. It was also shown that a possible alternative is to use the intrinisic mechanics of the manipulator such as "redundant" degrees of freedom to modulate its dynamic behavior, and again the distinction between impedance and admittance is fundamental.

An advantage of impedance control is that it permits a unified treatment of many aspects of manipulator control. Real mechanical interaction may be treated in the same framework as free (unconstrained) motions by using the concept of virtual interaction. Targets to be acquired and obstacles to be avoided are unified as different types of virtual interaction. This is important because often a single workpice is both target and obstacle. For example, a motor housing inside of which a bearing is to be mounted combines both the aspects of a target (the bearing must go inside it) and the aspects of an obstacle (collision with the housing must be avoided).[33] Path control,[35, 45] rate control,[47, 48] acceleration control,[29] force control,[23, 42] the hybrid combination of force and position control in orthogonal directions needed for dealing with pure kinematic constraints,[31] and stiffness control[40] may be considered in a single framework by regarding them as important special (or degenerate) causes of impedance control (e.g., position control: maximize impedance; force control: minimize impedance; stiffness control: linear, static impedance) but most important, the applicability of impedance control extends beyond the workless conditions imposed by free motions or pure kinematic constraints to include the control of energetic interactions such as are encountered when using a power tool.[14] Essentially, impedance control is an attempt to unify the control of "transport" tasks (which are the philosophical underpinning of conventional robot control) with the control of "interactive" tasks such as the use of tools.

The versatility of the approach stems from the fact that impedance is a fundamental aspect of manipulator behavior. It is a natural "language" for describing and communicating desired and achieved interactive behavior. The complete end-point impedance may be decomposed and because of the inevitable presence of an admittance (e.g. mass), these components may be recombined by simple superposition. Primitive components of a commanded manipulator behavior such as those represented by the potential functions of Fig. 3 may be combined by simple addition and the result is a member of the same family, another potential function (see Fig. 3); Thus impedances have some of the features desirable in a language. Potential functions also lend themselves to graphical representation and consequently, impedance control is naturally suited to both symbolic and graphical data bases and may facilitate the integration of Robotics and Computer Aided Design.

Impedance control promises to be particularly useful for understanding, controlling and coordinating the actions of mutually interacting manipulators, such as the fingers of a hand, the hand and the arm, or two arms. Using this approach each subsystem presents a simple behavior to the other subsystems. As a result, prediction and control of the combined behavior of the entire system is simplified. The ultimate goal of this effort is to understand the subtleties of adaptive tool-using, one of the distinguishing features of primate behavior. Impedance control may provide the basis of an understanding of tool-using behavior and permit its practical implementation on an amputee's artificial limb or on an industrial robot.

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APPENDIX I

Generalized inertial systems and the mobility tensor

Any mechanical linkage is a generalized inertial system. The defining property of an inertial system is its ability to store kinetic energy, defined as the integral of (generalized) velocity with respect to (generalized) momentum.[8] At any configuration defined by the generalized coordinates (e.g., the joint angles $\boldsymbol{\theta}$ for the linkages of Fig. 2) the kinetic energy is a quadratic form in (generalized) momentum.

$$E_k = \tfrac{1}{2}\,\mathbf{h}^t Y(\boldsymbol{\Theta})\,\mathbf{h}.$$

From Hamilton's equations,[43] the (generalized) velocity is the momentum gradient of the kinetic energy.

$$H(\mathbf{h},\boldsymbol{\Theta}) = E_k(\mathbf{h},\boldsymbol{\Theta})$$
$$\mathrm{d}\boldsymbol{\Theta}/\mathrm{d}t = \boldsymbol{\omega} = \nabla_h H = Y(\boldsymbol{\Theta})\mathbf{h}.$$

Kinetic energy is commonly confused with kinetic co-energy. The two are not identical and are related by a Legendre transform.[8]

$$E_k^* = \boldsymbol{\omega}^t\mathbf{h} - E_k = \boldsymbol{\omega}^t Y^{-1}\boldsymbol{\omega} - \tfrac{1}{2}\,\boldsymbol{\omega}^t Y^{-1} Y Y^{-1}\boldsymbol{\omega}$$

$$E_k^* = \tfrac{1}{2}\,\boldsymbol{\omega}^t Y^{-1}(\boldsymbol{\Theta})\,\boldsymbol{\omega} = \tfrac{1}{2}\,\boldsymbol{\omega}^t I(\Theta)\,\boldsymbol{\omega}.$$

At any configuration kinetic co-energy is a quadratic form in (generalized) velocity and its velocity gradient is the (generalized) momentum.[8]

$$\mathbf{h} = I(\boldsymbol{\Theta})\,\boldsymbol{\omega}.$$

For a generalized inertial system, Y is a symmetric, twice-contravariant tensor. To distinguish it from its inverse, the inertia tensor I, (symmetric, twice-convariant) Y will be termed the mobility tensor.[21]

A knowledge of the geometric relation between coordinate frames is sufficient to transform any tensor from one frame to another. As the joint angles are a set of generalized coordinates, for any configuration of the linkages of Fig. 2 the end-point coordinates are related to the joint angles via the kinematic transformations.

$$\mathbf{X} = L(\boldsymbol{\Theta}).$$

Differentiating these transformations yields the relation between velocities (at any given configuration).

$$\mathrm{d}\mathbf{X}/\mathrm{d}t = \mathbf{V} = J(\boldsymbol{\Theta})\,\boldsymbol{\omega}$$

$J(\boldsymbol{\Theta})$ in these equations is the configuration-dependent Jacobian. As the coordinate transformation does not store, dissipate or generate energy, incremental changes in energy are the same in all coordinate frames. This yields the relation between forces in each coordinate frame.

$$\mathrm{d}E_p = \mathbf{T}^t\mathrm{d}\boldsymbol{\Theta} = \mathbf{F}^t\mathrm{d}\mathbf{X} = \mathbf{F}^t J(\boldsymbol{\Theta})\mathrm{d}\boldsymbol{\Theta}.$$

At any given configuration.

$$\mathbf{T} = J^t(\boldsymbol{\Theta})\mathbf{F}.$$

The same approach yields the relation between the momenta in each coordinate frame.

$$dE_k = d\mathbf{h}^t\boldsymbol{\omega} = d\mathbf{p}^t\mathbf{V} = d\mathbf{p}^t J(\boldsymbol{\Theta})\boldsymbol{\omega}.$$

At any given configuration

$$\mathbf{h} = J^t(\boldsymbol{\Theta})\mathbf{p}.$$

These relations may be used to express the mobility in end-point coordinates.

$$\mathbf{V} = J\boldsymbol{\omega} = JY\mathbf{h} = JYJ^t\mathbf{p}.$$

Denoting the end-point mobility by $W(\boldsymbol{\Theta})$

$$W(\boldsymbol{\Theta}) = JYJ^t$$
$$\mathbf{V} = W(\boldsymbol{\Theta})\mathbf{p}.$$

The physical meaning of the mobility tensor is that if the system is at rest an applied force will produce an acceleration equal to the force vector premultiplied by the mobility tensor. At rest, $d\boldsymbol{\Theta}/dt = 0$ and hence:

$$d\mathbf{V}/dt = J d\boldsymbol{\omega}/dt$$
$$d\boldsymbol{\omega}/dt = Y d\mathbf{h}/dt.$$

From the generalized Hamilton[43]

$$d\mathbf{h}/dt = \mathbf{T} - \nabla_{\Theta}H.$$

At rest $\mathbf{h} = 0$, hence $H(\mathbf{h},\boldsymbol{\Theta}) = E_k = 0$ and $\nabla_{\Theta}H = 0$. Thus:

$$d\mathbf{H}/dt = \mathbf{T}$$
$$d\mathbf{V}/dt = JYJ^t\mathbf{F} = W\mathbf{F}.$$

As the mobility tensor is symmetric it may be diagonalized by rotating the coordinate axes to coincide with its eigenvectors. A force applied in the direction of an eigenvector (when the system is at rest) results in an acceleration in the same direction equal to the applied force multipled by the corresponding eigenvalue. The eigenvalues represent the inverse of the apparent mass or inertia seen by the applied force or torque.

Because the kinetic energy is a quadratic form in momentum, it may be represented graphically by an ellipsoid (see Fig. 5), the ellipsoid of gyration.[46] This may be thought of as the set of all momenta which produce the same kinetic energy (an isokinetic contour in momentum space). The lengths of the principle axes of the ellipsoid of gyration are inversely proportional to the square roots of the eigenvalues, proportional to the square roots of the associated apparent mass or inertia. The long direction of the ellipsoid of Fig. 5 is the direction of the greatest apparent inertia.

In the general case when the system is not at rest the relation between applied force and resulting motion is (in general) nonlinear and must be written in terms of a complete set of state equations for the inertial system. A convenient set of state variables are the Hamiltonian states, generalized position (e.g., $\boldsymbol{\Theta}$) and generalized momentum ($\mathbf{h}$). The state and output equations are in the form of a generalized admittance[21] as follows.

State equations:

$$d\mathbf{h}/dt = -\nabla_{\Theta}[\tfrac{1}{2}\, \mathbf{h}^t Y(\boldsymbol{\Theta})\mathbf{h}] + J^t(\boldsymbol{\Theta})\mathbf{F}$$
$$d\boldsymbol{\Theta}/dt = \nabla_h[\tfrac{1}{2}\, \mathbf{h}^t Y(\boldsymbol{\Theta})\mathbf{h}] = Y(\boldsymbol{\Theta})\mathbf{h}.$$

Output equations (position and velocity):

$$\mathbf{X} = L(\boldsymbol{\Theta})$$
$$\mathbf{V} = J(\boldsymbol{\Theta})Y(\boldsymbol{\Theta})\mathbf{h}.$$

APPENDIX II

A feedback law for impedance control

Assume that the desired end-point behavior to be imposed on the manipulator is given by:

$$M\, d\mathbf{V}/dt - D(\mathbf{V}_o - \mathbf{V}) - E(\mathbf{X}_o - \mathbf{X}) = \mathbf{F}_{int}.$$

Assume that an adequate model of the manipulator dynamics is:

$$I(\boldsymbol{\Theta})\, d\boldsymbol{\omega}/dt + C(\boldsymbol{\Theta},\boldsymbol{\omega}) + V(\boldsymbol{\omega}) + S(\boldsymbol{\Theta}) = \mathbf{T}_{act} + \mathbf{T}_{int}.$$

In this equation, $I(\boldsymbol{\Theta})$ is the configuration-dependent inertia tensor for the manipulator, $C(\boldsymbol{\Theta},\boldsymbol{\omega})$ are the inertial coupling terms (due to centrifugal and coriolis accelerations), $V(\boldsymbol{\omega})$ includes any velocity-dependent forces (e.g., frictional) and $S(\boldsymbol{\Theta})$ includes any static configuration-dependent forces (e.g., gravitational). Any actuator dynamics have been neglected and the actuator forces (or torques) $\mathbf{T}_{act}$ are assumed to be the control input to the manipulator.

The equation for the desired behavior may be regarded as a specification of the desired end-point acceleration which is to result from an external force impressed on the manipulator admittance.

$$d\mathbf{V}/dt = M^{-1}E(\mathbf{X}_o - \mathbf{X}) + M^{-1}D(\mathbf{V}_o - \mathbf{V}) + M^{-1}\mathbf{F}_{int}.$$

The corresponding acceleration in actuator coordinates is obtained by differentiating the kinematic transformations.

$$d\mathbf{V}/dt = J(\boldsymbol{\Theta})\, d\boldsymbol{\omega}/dt + G(\boldsymbol{\Theta},\boldsymbol{\omega})$$

where $G(\boldsymbol{\Theta},\boldsymbol{\omega}) = [d\{J(\boldsymbol{\Theta})\,\boldsymbol{\omega}\}/dt]\,\boldsymbol{\omega}$

$$d\boldsymbol{\omega}/dt = J^{-1}(\boldsymbol{\Theta})\,[d\mathbf{V}/dt - G(\boldsymbol{\Theta},\boldsymbol{\omega})].$$

Each of the impedance terms in the desired end-point behavior may be expressed in actuator coordinates using the kinematic transformations (see Appendix I).

$$E(\mathbf{X}_o - \mathbf{X}) = E[\mathbf{X}_o - L(\mathbf{\Theta})]$$
$$D(\mathbf{V}_o - \mathbf{V}) = D[\mathbf{V}_o - J(\mathbf{\Theta})\,\boldsymbol{\omega}].$$

For the purposes of controller design, each of these terms may be regarded as a component of a desired feedback law relating the control input $\mathbf{T}_{act}$ to measurable variables $\mathbf{\Theta}$, $\boldsymbol{\omega}$ and $\mathbf{F}_{int}$. The complete control law is obtained by substitution.

$$\mathbf{T}_{act} = I(\mathbf{\Theta})\,J^{-1}(\mathbf{\Theta})\,M^{-1}E[\mathbf{X}_o - L(\mathbf{\Theta})] + S(\mathbf{\Theta})$$
(position terms)
$$+ I(\mathbf{\Theta})\,J^{-1}(\mathbf{\Theta})\,M^{-1}D[\mathbf{V}_o - J(\mathbf{\Theta})\,\boldsymbol{\omega}] + V(\boldsymbol{\omega})$$
(velocity terms)
$$+ I(\mathbf{\Theta})\,J^{-1}(\mathbf{\Theta})\,M^{-1}\,\mathbf{F}_{ext} - J^{t}(\mathbf{\Theta})\mathbf{F}_{int}$$
(force terms)
$$- I(\mathbf{\Theta})\,J^{-1}(\mathbf{\Theta})\,G(\mathbf{\Theta},\boldsymbol{\omega}) + C(\mathbf{\Theta},\boldsymbol{\omega}).$$
(inertial coupling terms)

Note that although this equation does require the inverse Jacobian, it does not require the inverse kinematic transformation and only requires the forward kinematic equations. This is important in those cases in which no closed form for the inverse kinematic transformation exists.

APPENDIX III

Optimal impedance for a one-dimensional dynamic system

The system equations in phase variable form are:

$$\begin{bmatrix} \dot{X} \\ \dot{V} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} X \\ V \end{bmatrix} + \begin{bmatrix} 0 \\ k/m \end{bmatrix} X_o + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F_{ext}$$

The interface force is: $F = k(X_o - X) + bV$.

The objective function to be minimized is:

$$Q = \int_o^{\infty} \{ (F/F_{tol})^2 + [(X_o - X)/X_{tol}]^2 \}\,dt.$$

The external force F_{ext} is a zero-mean, Gaussian, purely random process of strength S. Thus:

$$E\,[F_{ext}(t)] = 0 \qquad E\,[F_{ext}(t)\,F_{ext}(t+\tau)] = S\delta(\tau).$$

In steady state $\overline{X} = X_o$, $\overline{V} = 0$ thus without loss of generality assume $X_o = 0$.

The covariance propagation equations are:

$$\dot{\overline{X^2}} = 2\overline{XV}$$
$$\dot{\overline{XV}} = V^2 - \overline{XV}\,b/m - \overline{X^2}\,k/m$$
$$\dot{\overline{V^2}} = S/m^2 - 2\overline{V^2}\,b/m - 2\overline{XV}\,k/m.$$

Because of the random forcing, the optimum impedance is obtained by minimizing the expectation of the objective function subject to the constraints imposed by the convariance propagation equations. Writing $p^2 = F_{tol}/X_{tol}$

$$E\,[Q] = \frac{1}{F_{tol}^2} \int_o^{\infty} \{b^2\,\overline{V^2} + 2k\,b\,\overline{XV} + (k^2 + p^2)\overline{X^2}\}\,dt.$$

The Pontryagin function is:

$$H = b^2\,\overline{V^2} + 2kb\,\overline{XV} + (k^2 + p^2)\,\overline{X^2} + 2\lambda_1\,\overline{XV}$$
$$+ \lambda_2(\overline{V^2} - \overline{XV}b/m - \overline{X^2}k/m)$$
$$+ \lambda_3(S/m^2 - 2\overline{V^2}b/m - 2\,\overline{XV}k/m).$$

The minimizing conditions are:

$$\frac{\partial H}{\partial k} = 0 = 2b\,\overline{XV} + 2k\overline{X^2} - \lambda_2\,\overline{X^2}/m - 2\lambda_3\,\overline{XV}/m.$$

$$\frac{\partial H}{\partial b} = 0 = 2b\,\overline{V^2} + 2k\overline{XV} - \lambda_2\,\overline{XV}/m - 2\lambda_3\,\overline{V^2}/m.$$

The LaGrange multipliers are determined from the co-state equations:

$$\frac{\partial H}{\partial \overline{X^2}} = -\dot{\lambda}_1 = (k^2 + p^2) - \lambda_2\,k/m.$$

$$\frac{\partial H}{\partial \overline{XV}} = -\dot{\lambda}_2 = 2kb + 2\lambda_1 - \lambda_2\,b/m - 2\lambda_3 k/m.$$

$$\frac{\partial H}{\partial \overline{V^2}} = -\dot{\lambda}_3 = b^2 + \lambda_2 - 2\lambda_3\,b/m.$$

Assuming a steady-state solution exists, it may be obtained by setting all rates of change to zero. Manipulating the resulting equations yields:

$$\overline{XV} = 0$$

$$\overline{V^2} = \frac{S}{2bm}$$

$$\overline{X^2} = \frac{S}{2bk}.$$

$$k_{opt}^2 = p^2$$

$$k_{opt} = F_{tol}/X_{tol}$$

$$b_{opt}^2 = 2k_{opt}\, m$$

$$b_{opt} = (\surd 2k_{opt} m).$$