Impedance Control: An Approach to Manipulation:

Part II—Implementation

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This three-part paper presents an approach to the control of dynamic interaction between a manipulator and its environment. Part I presented the theoretical reasoning behind impedance control. In Part II the implementation of impedance control is considered. A feedback control algorithm for imposing a desired cartesian impedance on the end-point of a nonlinear manipulator is presented. This algorithm completely eliminates the need to solve the "inverse kinematics problem" in robot motion control. The modulation of end-point impedance without using feedback control is also considered, and it is shown that apparently "redundant" actuators and degrees of freedom such as exist in the primate musculoskeletal system may be used to modulate end-point impedance and may play an essential functional role in the control of dynamic interaction.

Introduction

Most successful applications of industrial robots to date have been based on position control, in which the robot is treated essentially as an isolated system. However, many practical tasks to be performed by an industrial robot or an amputee with a prosthesis fundamentally require dynamic interaction. The work presented in this three-part paper is an attempt to define a unified approach to manipulation which is sufficiently general to control manipulation under these circumstances.

In Part I this approach was developed by starting with the reasonable postulate that no controller can make the manipulator appear to the environment as anything other than a physical system. An important consequence of dynamic interaction between two physical systems such as a manipulator and its environment is that one must physically complement the other: Along any degree of freedom, if one is an impedance, the other must be an admittance and vice versa.

One of the difficulties of controlling manipulation stems from the fact that while the bulk of existing control theory applies to linear systems, manipulation is a fundamentally nonlinear problem. The familiar concepts of impedance and admittance are usually applied to linear systems and regarded as equivalent and interchangeable. As shown in Part I, for a nonlinear system, the distinction between the two is fundamental.

Now, for almost all manipulatory tasks the environment at least contains inertias and kinematic constraints, physical systems which accept force inputs and which determine their motion in response and are properly described as admittances. When a manipulator is mechanically coupled to such an environment, to ensure physical compatibility with the environmental admittance, something has to give, and the manipulator should assume the behavior of an impedance.

Thus a general strategy for controlling a manipulator is to control its motion (as in conventional robot control) and in addition give it a "disturbance response" for deviations from that motion which has the form of an impedance. The dynamic interaction between manipulator and environment may then be modulated, regulated, and controlled by changing that impedance.

This second part of the paper presents some techniques for controlling the impedance of a general nonlinear multiaxis manipulator.

Implementation of Impedance Control

A distinction between impedance control and the more conventional approaches to manipulator control is that the controller attempts to implement a dynamic relation between manipulator variables such as end-point position and force rather than just control these variables alone. This change in perspective results in a simplification of several control problems.

Most of our work to date [3, 6, 13, 14, 16] has focused on controlling the impedance of a manipulator as seen at its "port of interaction" with the environment, its end effector. A substantial body of literature has been published on methods for implementing a planned end effector cartesian path [5, 27, 28, 32, 34, 35]. The approach is widely used in the control of industrial manipulators and there is some evidence of a comparable strategy of motion control in biological systems [1, 24]. Following the lead from this prior work we have investigated ways of presenting the environment with a dynamic behavior which is simple when expressed in workspace (e.g., Cartesian) coordinates.

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The lowest-order term in any impedance is the static relation between output force and input displacement, a stiffness. If, in common with much of the current work on robot control, we assume actuators capable of generating commanded forces (or torques), Tact, sensors capable of observing actuator position (or angle), θ, and a purely kinematic relation (i.e., no structural elastic effects) between actuator position and end-point position[1], $\mathbf{X}=L(\theta)$, it is straightforward to design a feedback control law to implement in actuator coordinates a desired relation between end-point (interface) force, Fint, and position, $\mathbf{X}$. Defining the desired equilibrium position for the end-point in the absence of environmental forces (the virtual position) as $\mathbf{X}_0$, a general form for the desired force-position relation is:

$$\mathrm{Fint}=K[\mathbf{X}_0-\mathbf{X}] \tag{1}$$

Compute the Jacobian, $\mathbf{J}(\theta)$:

$$d\mathbf{X}=\mathbf{J}(\theta)d\theta \tag{2}$$

From the principal of virtual work:

$$\mathrm{Tact}=\mathbf{J}^t(\theta)\mathrm{Fint} \tag{3}$$

The required relation in actuator coordinates is:

$$\mathrm{Tact}=\mathbf{J}^t(\theta)K\,[\mathbf{X}_0-L(\theta)] \tag{4}$$

No restriction of linearity has been placed on the relation $K[\mathbf{X}_0-\mathbf{X}]$, and the displacement of $\mathbf{X}$ from $\mathbf{X}_0$ need not be small. Note that in this equation the inverse Jacobian is not required.

Inverting the kinematic equations of a manipulator to determine the time-history of actuator (joint) positions required to produce a desired time-history of end-point positions has been described as one of the most difficult problems in robot control [28]. For some manipulators (e.g., those with nonintersecting wrist joint axes) no explicit (closed-form) algebraic solution may be possible. However, if $K[\mathbf{X}_0-\mathbf{X}]$ is chosen so as to make the end-point sufficiently stiff, then a controller which implements equation (4) will accomplish Cartesian end-point position control and the need to solve the "inverse kinematics problem" has been completely eliminated. Only the forward kinematic equations for the manipulator need be computed. This is a direct consequence of the care which was taken to ensure that the desired behavior was compatible with the fundamental mechanics of manipulation and was expressed as an impedance.

Another important term in the manipulator impedance is the relation between force and velocity. Again, given the above assumptions, it is straightforward to define a feedback law to implement in actuator coordinates a desired relation between end-point force and end-point velocity such as:

$$\mathrm{Fint}=B[\mathbf{V}_0-\mathbf{V}] \tag{5}$$

From the manipulator kinematics:

$$\mathbf{V}=\mathbf{J}(\theta)\omega \tag{6}$$

The required relation in actuator coordinates is:

$$\mathrm{Tact}=\mathbf{J}^t(\theta)B[\mathbf{V}_0-\mathbf{J}(\theta)\omega] \tag{7}$$

Again note that the relation $B[\mathbf{V}_0-\mathbf{V}]$ need not be linear and that inversion of the Jacobian is not required.

The dynamic behavior to be imposed on the manipulator should be as simple as possible, but no simpler. The foregoing equations take no account of the inertial, frictional, or gravitational dynamics of the manipulator. Under some circumstances this may be reasonable, but in many situations these effects cannot be neglected. To ensure dynamic feasibility the choice of the impedance to be imposed should be based on the dominant dynamic behavior of the manipulator. The choice is a tradeoff between keeping the complexity of the controller within manageable limits while ensuring that imposed behavior adequately reflects the real dynamic behavior of the controlled system. As a result it depends both on the manipulator itself and on the environment in which it operates. For example, a manipulator intended for underwater applications will operate in a predominantly viscous environment and it may be reasonable to ignore inertial effects. In contrast, a manipulator intended for operation in a free-fall orbit will encounter a predominantly inertial environment. For terrestrial applications (which have been the main concern of our work) both gravitational and inertial effects are important, and the dominant dynamic behavior is that of a mass driven by motion-dependent forces, second order in displacement along each degree of freedom.

[1] Throughout this paper, "position" will refer to both location and orientation, and "force" will refer to both force and moment.

Nomenclature

- Y = admittance
- Z = impedance
- $:\{c\}$ = modulation by command set
- Sf = flow source
- Se = effort source
- Fext = external force
- Fint = interface force
- $\mathbf{X}$ = end-point position
- $\mathbf{X}_0$ = commanded (virtual) position
- $\mathbf{V}$ = end-point velocity
- $\mathbf{V}_0$ = commanded (virtual) velocity
- $K[\bullet]$ = force/displacement relation
- $B[\bullet]$ = force/velocity relation
- θ = actuator position or angle
- ω = actuator velocity
- $L(\bullet)$ = linkage kinematic equations
- $\mathbf{J}(\theta)$ = Jacobian
- M = inertia tensor in end-point coordinates
- m = mass
- I = inertia
- t = time
- $F(\bullet)$ = noninertial impedance
- Me = environmental inertia tensor
- $I(\theta)$ = inertia tensor in actuator coordinates
- $C(\theta,\omega)$ = inertial coupling torques
- $V(\omega)$ = velocity-dependent torques
- $S(\theta)$ = position-dependent torques
- $G(\theta,\omega)$ = accelerative coupling terms
- Tact = actuator force or torque
- Tint = interface torques
- $Y(\theta)$ = mobility tensor in actuator coordinates
- $W(\theta)$ = mobility tensor in end-point coordinates
- $\mathbf{h}$ = generalized momentum in actuator coordinates
- $\mathbf{p}$ = generalized momentum in end-point coordinates
- $H(\bullet)$ = Hamiltonian
- $\mathbf{T},T_1,T_2$ = torque
- θ_1,θ_2 = absolute joint angle
- ρ_1,ρ_2 = relative joint angle
- L_1,L_2 = link lengths
- Ke = net stiffness due to elbow muscles
- Ks = net stiffness due to shoulder muscles
- Kt = net stiffness due to two-joint muscles
- Kx = stiffness tensor in end-point coordinates
- Ko = stiffness tensor in joint coordinates
- Ep = potential energy
- Ek = kinetic energy
- Ek^* = kinetic coenergy
- $\lambda_1\lambda_2$ = eigenvalues

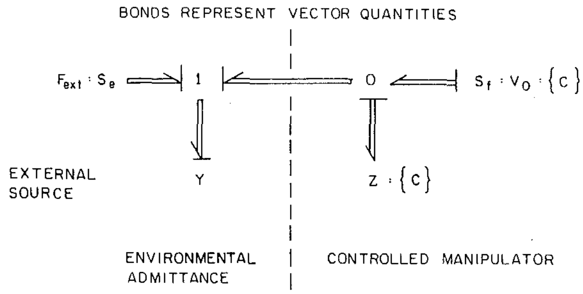

Fig. 1 A bond graph equivalent network representation of an impedance-controlled manipulator interacting with an environmental admittance. Each bond represents a vector of power flows along multiple degrees of freedom.

When the manipulator is decoupled from its environment the term due to the environmental admittance disappears, and in principle the manipulator alone need exhibit no mass-like behavior. In practice, the uncoupled manipulator still has inertia (albeit nonlinear and configuration-dependent). This means that the controlled system, both with the manipulator coupled to and uncoupled from its environment, can be represented by an admittance coupled to an impedance as shown in Fig. 1.

No physically realisable strategy can eliminate the inertial effects of a manipulator but the apparent inertia seen at the end effector can be modified. The approach we have taken to deal with inertial manipulator behavior is to "mask" the true nonlinear inertial dynamics of the manipulator and impose simpler dynamics, those of a rigid body. Most manipulatory tasks are fundamentally described in relative coordinates, that is, in terms of displacements and rotations with respect to a workpiece, tool or fixture whose location in the workspace is not known in advance with certainty. As a result, task planning and execution will be simplified if the end-point inertial behavior is modified to be that of a rigid body with an inertia tensor which remains invariant under translation and rotation of the coordinate axes. This is achieved if:

$$M = \begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{8}$$

This is the inertia tensor of a rigid body such as a cube of uniform density. This inertia tensor eliminates the angular velocity product terms in the Euler equations for the motion of a rigid body [8] and ensures that if the system is at rest the applied force and the resulting acceleration vectors are colinear.

To represent the dominant second-order behavior of the manipulator the noninertial interface forces are assumed to depend only on displacement, velocity and time:

$$\mathbf{F}\mathrm{int} = F(\mathbf{X},\mathbf{V}) - Md\mathbf{V}/dt \tag{9}$$

If the noninertial behavior to be imposed is nodic, it may be written in terms of a displacement from a commanded (time-varying) position $\mathbf{X}_0$:

$$\mathbf{F}\mathrm{int} = F(\mathbf{X}_0 - \mathbf{X}, \mathbf{V}_0 - \mathbf{V}) - Md\mathbf{V}/dt \tag{10}$$

Although there may be cases in which coupled nonlinear viscoelastic behavior is useful, for simplicity the position- and velocity-dependent terms may be separated:

$$\mathbf{F}\mathrm{int} = K[\mathbf{X}_0 - \mathbf{X}] + B[\mathbf{V}_0 - \mathbf{V}] - Md\mathbf{V}/dt \tag{11}$$

All of the parameters in this expression are implicitly assumed to be functions of the set of control commands $\{c\}$ and of time.

This set of assumptions defines a target behavior which includes inertial effects. The first two terms are the position- and velocity-dependent impedances of equations (1) and (5). If the environment is a simple rigid body acted on by unpredictable (or merely unpredicted) forces, its dynamic equations are:

$$Me\, d\mathbf{V}/dt = \mathbf{F}\mathrm{ext} + \mathbf{F}\mathrm{int} \tag{12}$$

and the coupled equations of motion for the complete system of figure 1 are:

$$(Me + M)d\mathbf{V}/dt = K[\mathbf{X}_0 - \mathbf{X}] + B[\mathbf{V}_0 - \mathbf{V}] + \mathbf{F}\mathrm{ext} \tag{13}$$

Note that in this case both the coupled and uncoupled equations for the system have the same second-order form.

To implement the target behavior of equation (11), one approach we have used is to express the desired Cartesian coordinate impedance in actuator coordinates (the kinematic transformations between actuator coordinates and end-point coordinates provides sufficient information to do this) and then use a model of the manipulator dynamics to derive the required controller equations. Assuming that the kinematic, inertial, gravitational, and frictional effects provide an adequate model of the manipulator dynamics as follows:

$$I(\theta)d\omega/dt + C(\theta,\omega) + V(\omega) + S(\theta) = \mathbf{T}\mathrm{act} + \mathbf{T}\mathrm{int} \tag{14}$$

an expression for the required actuator torque as a function of actuator position and velocity and end-point force can be derived by straightforward substitution (see Appendix I):

$$\begin{aligned} \mathbf{T}\mathrm{act} &= I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}K[\mathbf{X}_0 - L(\theta)] + S(\theta) \\ &+ I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}B[\mathbf{V}_0 - \mathbf{J}(\theta)\omega] + V(\omega) \\ &+ I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}\mathbf{F}\mathrm{int} - \mathbf{J}^{t}(\theta)\mathbf{F}\mathrm{int} \\ &- I(\theta)\mathbf{J}^{-1}(\theta)G(\theta,\omega) + C(\theta,\omega) \end{aligned} \tag{15}$$

This equation expresses the required behavior to be provided by the controller as a nonlinear impedance in actuator coordinates. It may be viewed as a nonlinear feedback law relating actuator torques to observations of actuator position, velocity and interface force. The input (command) variables are the desired cartesian position (and velocity) and the terms of the desired (possibly nonlinear) cartesian dynamic behavior characterized by M, $B[\cdot]$ and $K[\cdot]$.

The feasibility of this approach to cartesian impedance control has been investigated [6,16] by implementing this nonlinear control law to impose cartesian end-point dynamics on a servo-controlled, planar, two-link mechanism (similar to the nonlinear linkage in a SCARA[2] robot). A simple analysis estimating the computation required to implement this controller on a six-degree-of-freedom manipulator indicated that the computational burden is comparable to "exact" approaches to generating forward-path manipulator commands such as the recursive LaGrangian [17] and Newton-Euler [21] methods or the configuration space method [18].

If the interface forces and torques in equations (11) and (15) are eliminated and the position- and velocity-dependent terms reduced to linear diagonal forms, this implementation of impedance control resembles the resolved acceleration method [22]. However, unlike the resolved acceleration method, the impedance control algorithm presented above is based on desired end-point behaviour which may be chosen rationally using approaches such as the optimisation technique presented in Part III. Furthermore, the impedance control algorithm includes terms for coping with external "disturbances." Without the external "disturbance" terms (which have no counterpart in the resolved acceleration algorithm) the manipulator is not capable of controlled

[2] Selective Compliance Assembly Robot Arm [23].

mechanical interaction with its environment. Note also that the above approach to defining the controller equations is not restricted to commanded linear behavior and can be applied equally well to achieve the more general coupled nonlinear behavior of equation (9).

It is not claimed that the above algorithm is the only way to achieve a desired end-point impedance. It is presented here only to demonstrate that a control law capable of modulating the end-point impedance of a manipulator may be formulated. The controller of equation (15) was designed by a technique which is similar to pole-placement methods [31] in that the desired behaviour and a model of the actual behaviour of the manipulator were compared algebraically to derive the controller equations. In common with most approaches to manipulator control the approach is based on a model which ignored many aspects of real manipulator performance, particularly the dynamics of the actuators and the transmission system. Furthermore, like many other approaches the method assumes that the Jacobian is invertible.

This technique is, of course, only one possible approach to the design of a controller for implementing a desired cartesian impedance, and, if one may draw from linear systems design experience without overstretching the analogy to pole-placements methods, it is not even likely to be the best. Other approaches to controller design such as the model-referenced adaptive control method [9] will probably be useful.

Impedance Modulation Without Feedback

Modulation of end-point impedance using feedback strategies is not the only way to control the dynamic behavior of a manipulator, nor is it always the best. This is particularly evident in a biological system. One of the most distinctive features of the primate neural control system is the unavoidable delay associated with neural transmission. The *shortest* time for information to get from peripheral sensors (e.g., in the muscles or skin) in the human arm to the higher levels of the central nervous system (e.g., the cortex) and back to the actuators of the arm is 70 milliseconds, and loop transmission delays of 100 to 150 milliseconds are typical [29]. This problem is further exacerbated if significant computation is required (the response time to a visual stimulus is somewhere between 200 and 250 milliseconds). The effectiveness of feedback control in the presence of a delay of this magnitude is severely limited, particularly in dealing with tasks involving dynamic interaction. Yet primates excel at controlling dynamic interactions; How do they do that?

One alternative to feedback which we have explored is the use of redundancies: "excess" actuators or "excess" skeletal degrees of freedom. From a purely kinematic standpoint the neuromuscular system is multiply redundant. For example, the kinematic chain connecting the wrist joint to the chest (clavicle, scapula, humerus, radius and ulna) has considerably more degrees of freedom than those required to specify the position (and orientation) of the hand in cartesian coordinates. These skeletal redundancies can serve to provide a measure of control over the inertial component of the end-point dynamics.

In considering the apparent inertial behaviour of the end-point it is useful to remember that an inertia is fundamentally an admittance; flow (velocity) is determined as a response to impressed effort (force). Dealing with kinematic redundancy is considerably simplified if the constitutive equations are written as a relation determining generalised velocity, ω, (e.g., the velocities of the manipulator joints) as a function of generalized momentum, $\mathbf{h}$:

$$\omega = Y(\theta)\mathbf{h} \tag{16}$$

$Y(\theta)$ is the inverse of the more commonly used inertia tensor, and to help distinguish the two, the term "mobility" is suggested. The elements of the mobility tensor in general will depend on the manipulator configuration.

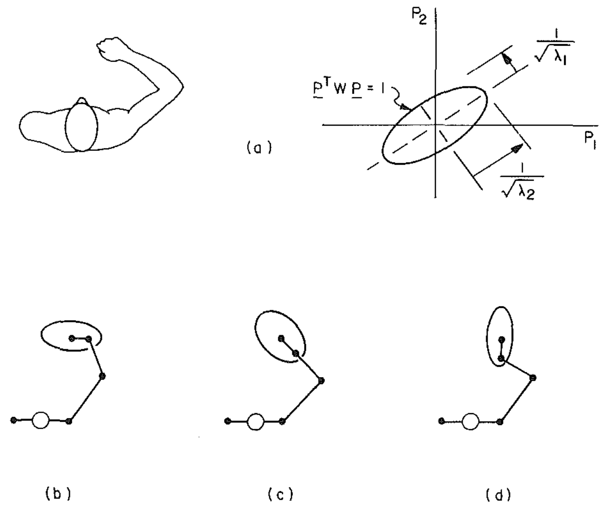

Fig. 2 A schematic representation of the influence of kinematic redundancies on the mobility (inverse effective mass) of the end-point of a planar linkage. The ellipsoid of gyration associated with the mobility tensor is shown in (*a*). The eigenvalues of the mobility tensor are inversely proportional to the effective mass in the direction of the corresponding eigenvectors and the square root of their ratio determines the ratio of the major and minor axes of the ellipsoid, which are colinear with the eigenvectors. For a planar, three-member linkage with links of uniform density and cross section and lengths in the ratio 1: 2: 3 the effect on the ellipsoid of gyration of changing the linkage configuration for a fixed position of the end-point is shown in (*b*), (*c*), and (*d*).

At any given configuration, the kinematic transformations between joint angles and end-point coordinates define not only the relations between generalized displacements, flows and efforts in the two coordinate frames, (see equations (2), (3), and (6)) they also define the relation between the generalised momenta in joint coordinates, $\mathbf{h}$, and end-point coordinates, $\mathbf{p}$, through the Jacobian (see Appendix II):

$$\mathbf{h} = \mathbf{J}^t(\theta)\mathbf{p} \tag{17}$$

Consequently, the mobility tensor in end-point coordinates $W(\theta)$ is related to the mobility in joint coordinates $Y(\theta)$ as follows:

$$\mathbf{V} = W(\theta)\mathbf{p} \tag{18}$$

$$W(\theta) = \mathbf{J}(\theta)Y(\theta)\mathbf{J}^t(\theta) \tag{19}$$

The physical meaning of the end-point mobility tensor is that if the system is at rest (zero velocity) then a force vector applied to the end-point causes an acceleration vector (not necessarily co-linear with the applied force) which is obtained by premultiplying the force vector by the mobility tensor (see Appendix II).

Note that the Jacobian in the above equation need not be square, and that the end-point mobility is configuration dependent. As a result, redundant degrees of freedom can be used to modulate the end-point mobility. Consider the simplified three-link model of the primate upper extremity (arm, forearm and hand, each considered to be rigid bodies, linked by simple pin-joints) moving in a plane as shown in Fig. 2. For simplicity, assume the links are rods of uniform density with lengths in the ratio of 1: 2: 3.

Any real linkage such as the skeleton is a generalised kinetic energy storage system. Kinetic energy is always a quadratic form in momentum:

$$Ek = \tfrac{1}{2}\mathbf{h}^t Y(\theta)\mathbf{h} \tag{20}$$

Thus the locus of deviations of the generalised momentum from zero for which the kinetic energy is constant is an ellipsoid, the "ellipsoid of gyration" [33]. It graphically

Table 1 Variation of apparent end-point mass with linkage configuration

Distal link orientation (degrees)	Effective mass X_1-direction (kgm)	Effective mass X_2-direction (kgm)
90	0.322	1.823
135	0.568	0.568
180	1.824	0.323

Link Lengths: 1, 2, 3 meters; Linear density: 1 kgm/m

represents the directional properties of the mobility tensor. The eigenvalues of the symmetric mobility tensor define the size and shape and the eigenvectors the orientation of the ellipsoid of gyration (see Appendix II). An ellipsoid of gyration can be associated with the mobility tensor in any coordinate frame, e.g., end-point coordinates (see Fig. 2(*a*)).

Figures 2(*b*) through 2(*d*) show the profound effect on the ellipsoid of gyration of changes in arm configuration while keeping the position of the end-point fixed. The inertial resistance to a force applied radially inward toward the shoulder (vertically downward in the figure) changes by almost a factor of six as the hand rotates through ninety degrees (see Table 1). In the configuration of Fig. 2(*d*) the applied force has to accelerate all three links; in that of Fig. 2(*b*) it primarily has to accelerate the distal link. Clearly, kinematic redundancies in a linkage provide a vehicle for changing the way the end-point will react to external disturbances without recourse to feedback strategies.

As an aside, an alternative representation of inertial behavior is via the ellipsoid of inertia [33]. Asada [4] has suggested its use as a tool for designing robot mechanisms. However, the ellipsoid of gyration is the more fundamental representation; it is readily obtained even when the Jacobian of the linkage is noninvertible. Also, while the matrix $Y(\theta)$ may never have zero eigenvalues, (assuming real links with nonzero mass) the matrix $W(\theta)$ may, because of the kinematics of the linkage. If the inertial behavior of the tip is written in the conventional (impedance) form:

$$\mathbf{p} = M(\theta)\mathbf{V} \tag{21}$$

there exist locations in the workspace for which the eigenvalues of the tensor $M(\theta)$ become infinite. Thus the end-point inertia tensor can not be defined for some linkage configurations. On the other hand the worst the eigenvalues of $W(\theta)$ will do is go to zero, which is easier to deal with computationally. Again, a reminder of the fact that the difference between impedance and admittance is fundamental.

Impedance Modulation Using Actuator Redundancies

It is also possible to modulate the position- and velocity-dependent components of end-point impedance without feedback by exploiting the intrinsic properties of the actuators, and again apparent redundancies are useful. Although a muscle is by no means thermodynamically conservative, it exhibits a static relation between force and length (for any given fixed level of neural input) similar to that of a mechanical spring, i.e., one which permits the definition of a potential function analogous to elastic energy.[3] Muscle force also exhibits a dependence on velocity similar to a mechanical damper. It has been shown that the mechanical impedance of a single muscle may be modulated by neural commands both in the presence and in the absence of neural feedback [7, 11, 12, 25, 26]. Simultaneously activating two or more muscles which oppose each other across a joint is one strategy which permits impedance to be modulated independent of joint torque [15, 20]. (This is what happens, for example, when one tenses the muscles of the arm without moving; the impedance of the limb increases.)

[3] Curiously, the force/length behaviour of most muscles is such that the co-energy integral is not defined and thus no compliance form is definable [29]: Muscles are impedances, not admittances.

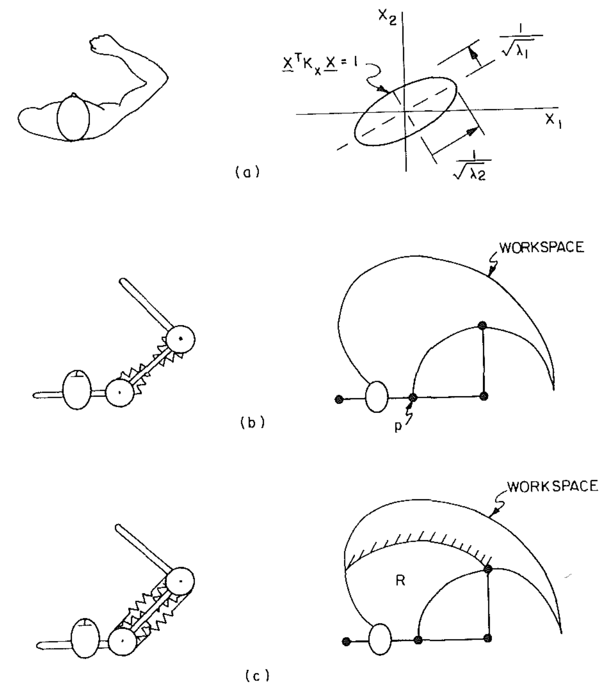

Fig. 3 A schematic representation of the influence of the polyarticular muscles of the primate upper extremity on the range of end-point stiffnesses which may be achieved without recourse to feedback strategies by simultaneous activation of opposing muscles. The ellipsoid associated with the symmetric differential stiffness tensor is shown in (*a*). The eigenvalues of the stiffness tensor are proportional to the stiffness in the direction of the corresponding eigenvectors and the square root of their ratio determines the ratio of the major and minor axes of the ellipsoid, which are colinear with the eigenvectors. Assuming the upper extremity may be modelled as a two-member linkage with equal link lengths, without biarticular muscles, a *necessary condition to achieve an end-point stiffness with equal* eigenvalues (hence a circular ellipsoid) is only satisfied at the point *p* on the workspace boundary as shown in (*b*). With biarticular muscles acting at equal moment arms about each joint an end-point stiffness with equal eigenvalues and a circular ellipsoid may be achieved throughout the region *R* shown in (*c*).

There are also considerably more skeletal muscles than joints, even beyond the antagonist pairing required to permit unidirectional muscle force to produce bidirectional joint torques. For example, the torque flexing the elbow joint (one of the simpler joints in the primate upper extremity) is generated by brachialis, brachioradialis, biceps capitus brevis, and biceps capitus longus. Does this complexity serve any purpose? If the control of end-point impedance of the limb without feedback is considered it will seen that these apparent actuator redundancies may have a functional role to play [13].

Consider the simplified two-link model of the primate upper limb (forearm and hand treated as a single rigid body, pin-jointed to the upper arm) moving in a horizontal plane as shown in Fig. 3. In the absence of feedback, the static component of the total end-point impedance will solely be due to the spring-like properties of the individual muscles. For each muscle, a potential function may be defined, and the combined effect of multiple muscles is to define a total potential function (which could be determined by adding the potential functions of the individual muscles). The total potential at any point is invariant under coordinate transformations and the total potential function may be expressed in any coordinate system by direct substitution.

Now, for simplicity, assume that the relations between muscle force and length and muscle length and joint rotation result in a linear torque/angle relation for each muscle. First consider the monoarticular (single-joint) muscles which generate torques about only a single joint: their combined effect is to define a diagonal stiffness tensor in relative angular coordinates:

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} Ks & 0 \\ 0 & Ke \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \tag{22}$$

Each of the terms Ks and Ke may vary. For example, the stiffness about the human elbow can vary from about 1 Nm/rad. to more than 200 Nm/rad [20, 36].

When this stiffness tensor is expressed in end-point coordinates, because of the distortion due to the nonorthogonality of the kinematic transformations the end-point stiffness will no longer be diagonal, but the range of end-point stiffnesses which could be achieved without feedback using monoarticular muscles to change Ks and Ke is quite restricted. This is readily seen in the shape of the potential function corresponding to this stiffness. For small displacements the potential function is a quadratic form and its isopotential contours are ellipsoids which graphically represent the directional character of the stiffness tensor (see Fig. 3(a)).

To illustrate the nature of the problem, suppose it were desired to have the end-point equally stiff in all directions. This would correspond to a potential function with circular isopotentials. However, given only single joint muscles, throughout the useful workspace a potential function with circular isopotentials can not be achieved. For example, assuming links of equal length and joint ranges of 0 to 90 degrees for the shoulder and 0 to 180 degrees for the elbow, a necessary condition to achieve circular isopotentials is only satisfied at one point (point p in Fig. 3(b)) on the boundary of the workspace (see Appendix III). This is because to specify a symmetric second-rank tensor such as stiffness in two dimensions requires three parameters and the monoarticular muscles provide only two.

However, the biomechanical system abounds with polyarticular muscles – muscles which generate torques about more than one joint. The biceps and triceps muscles of the upper arm cross both the elbow joint and the shoulder joint and provide a mechanical coupling between shoulder and elbow rotations which radically increases the range of stiffnesses which may be achieved without feedback.

For simplicity assume the same linear relation between muscle-generated torque and angle for both joints. Now, including the two-joint muscles, the stiffness tensor in relative joint angle coordinates will have off-diagonal terms:

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} Ks+Kt & Kt \\ Kt & Ke+Kt \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \tag{23}$$

The term Kt represents the contribution of the two-joint muscles and, like Ke and Ks, it may vary. Now suppose again that it is desired to have the end-point equally stiff in all directions. As a result of the two-joint muscles, as shown in Appendix III, a potential field with circular isopotentials could be achieved without feedback (by varying Ke, Ks, and Kt) throughout a much larger region in the workspace (region R in Fig. 3(c)). In effect, the two-joint muscles provide a third parameter with which to modulate the stiffness tensor. Note that this is not peculiar to the specific set of simplifying assumptions made above: In general, the availability of polyarticular muscles dramatically increases the range of end-point impedances which could be achieved without feedback.

The point of this discussion is to demonstrate that impedance control is possible without depending on feedback strategies, by using to advantage the intrinsic behavior of the manipulator "hardware." Apparent redundancies in the musculoskeletal system, which are frequently seen as presenting a coordination problem which the biological controller has to solve, may in fact represent a solution to a problem: they may play a functional role in controlling the interaction between the limb and the environment during dynamic events sufficiently rapid to limit the effectiveness of feedback control.

Summary

In this part of the paper, techniques for implementing a desired impedance on a manipulator were considered. Feedback control algorithms for imposing Cartesian impedances up to second order on a general nonlinear manipulator were presented. Because care was taken to ask for a manipulator behaviour which is compatible with the fundamental mechanics of manipulation, (as outlined in Part I) the need to solve the "inverse kinematics problem" – generally regarded as fundamental to all robot control – was circumvented.

Techniques for modulating the end-point impedance of a manipulator without recourse to feedback were also discussed. Multiple actuators and "excess" linkage degrees of freedom may also be used to modulate end-point impedance and it is suggested that the apparent redundancies in the primate musculoskeletal system may in fact play an essential functional role in controlling interactive behavior. The hypothesis that impedance modulation is one of the prominent strategies of natural movement control provides the motivation for a research project to develop a cybernetically controlled prosthesis which will give an amputee the ability to change its impedance at will [2].

The modulation of end-point impedance without feedback may also be important for industrial robots. Feedback loop transmission delays are not just a biological problem; It is widely recognized that computation time is one of the limiting factors in the design of robot controllers. It could be argued that as computation becomes cheaper and faster, this problem will disappear, but one reasonable way of describing manipulation is as a series of "collisions" with objects in the environment [10]. During a collision dynamic events take place extremely rapidly and *any* feedback controller may encounter difficulties. Control of dynamic interaction without feedback is an interesting alternative and is currently under investigation [19].

A feature of impedance control is that different controller actions (aimed at satisfying different task requirements) may be superimposed. For example, suppose that a desired end-point position- and velocity-dependent behaviour is implemented on a manipulator using a feedback control strategy as outlined above in equations (4) and (7). At the same time kinematic redundancies in the manipulator are used to modulate the end-point mobility. At any given end-point position, $\mathbf{X}$, (which is determinable from the configuration, θ) the manipulator configuation may be chosen to best approximate a desired inertial behaviour (for example, the mobility normal to a kinematic constraint surface may be maximised). This configuration may then be used in the feedback law which implements the position- and velocity-dependent behaviour. As the equations never require inversion of the Jacobian, they can be applied to a manipulator with kinematic redundancies. Note that this approach to end-point control in the presence of kinematic redundancies is significantly different from the use of a generalised pseudoinverse [35].

Part III of this paper will discuss the application of impedance control.

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APPENDIX I

A Nonlinear Feedback Law for Impedance Control

Assume that the desired end-point behavior to be imposed on the manipulator is given by:

$$M d\mathbf{V}/dt - B[\mathbf{V}_0 - \mathbf{V}] - K[\mathbf{X}_0 - \mathbf{X}] = \mathbf{F}\mathrm{int}$$

Assume that an adequate model of the manipulator dynamics is:

$$I(\theta)d\omega/dt + C(\theta,\omega) + V(\omega) + S(\theta) = \mathbf{T}\mathrm{act} + \mathbf{T}\mathrm{int}$$

In this equation, $I(\theta)$ is the configuration-dependent inertia tensor for the manipulator, $C(\theta,\omega)$ are the inertial coupling terms (due to centrifugal and coriolis accelerations), $V(\omega)$ includes any velocity-dependent forces (e.g., frictional) and $S(\theta)$ includes any static configuration-dependent forces (e.g., gravitational). Any actuator dynamics have been neglected. The actuator forces (or torques) **T**act are assumed to be the control input to the manipulator.

The equation for the desired behavior may be regarded as a specification of the desired end-point acceleration which is to result from an external force impressed on the manipulator admittance.

$$d\mathbf{V}/dt = M^{-1}K[\mathbf{X}_0 - \mathbf{X}] + M^{-1}B[\mathbf{V}_0 - \mathbf{V}] + M^{-1}\mathbf{F}\mathrm{int}$$

The corresponding acceleration in actuator coordinates is obtained by differentiating the kinematic transformations.

$$d\mathbf{V}/dt = \mathbf{J}(\theta)d\omega/dt + G(\theta,\omega)$$

where

$$G(\theta,\omega) = [d\{\mathbf{J}(\theta)\omega\}/d\theta]\omega$$

$$d\omega/dt = \mathbf{J}^{-1}(\theta)[d\mathbf{V}/dt - G(\theta,\omega)]$$

Each of the impedance terms in the desired end-point behavior may be expressed in actuator coordinates using the kinematic transformations

$$K[\mathbf{X}_0 - \mathbf{X}] = K[\mathbf{X}_0 - L(\theta)]$$

$$B[\mathbf{V}_0 - \mathbf{V}] = B[\mathbf{V}_0 - \mathbf{J}(\theta)\omega]$$

For the purposes of controller design, each of these terms may be regarded as a component of a desired feedback law relating the control input **T**act to the variables θ, ω and **F**int, which are assumed to be accessible measurements. The complete control law is obtained by substitution.

$$\begin{aligned} \mathrm{Tact} = {} & I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}K[\mathbf{X}_0 - L(\theta)] + S(\theta) \text{ (position terms)} \\ & + I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}B[\mathbf{V}_0 - \mathbf{J}(\theta)\omega] + V(\omega) \text{ (velocity terms)} \\ & + I(\theta)\mathbf{J}^{-1}(\theta)M^{-1}\mathrm{Fint} - \mathbf{J}^t(\theta)\mathrm{Fint} \text{ (force terms)} \\ & - I(\theta)\mathbf{J}^{-1}(\theta)G(\theta,\omega) + C(\theta,\omega) \text{ (inertial coupling terms)} \end{aligned}$$

Note that although this equation does require the inverse Jacobian, it does not require inversion of the kinematic equations. Only the forward kinematic equations need be computed. This will be important for those manipulators for which no explicit algebraic (closed form) solution to the inverse kinematic equations exists.

APPENDIX II

Generalized Inertial Systems and the Mobility Tensor

Any mechanical linkage is a generalized inertial system. The defining property of an inertial system is its ability to store kinetic energy, defined as the integral of (generalized) velocity with respect to (generalized) momentum [8]. At any configuration defined by the generalized coordinates the kinetic energy is a quadratic form in (generalized) momentum.

$$Ek = \tfrac{1}{2}\mathbf{h}^t\, Y(\theta)\mathbf{h}$$

From Hamilton's equations [30], the (generalized) velocity is the momentum gradient of the kinetic energy.

$$H(\mathbf{h},\theta) = Ek(\mathbf{h},\theta)$$

$$d\theta/dt = \omega = \nabla_{\mathbf{h}} H = Y(\theta)\mathbf{h}$$

Kinetic energy is commonly confused with kinetic coenergy. The two are not identical and are related by a Legendre transform [8].

$$Ek^* = \omega^t\mathbf{h} - Ek = \omega^t Y^{-1}\omega - \tfrac{1}{2}\omega^t Y^{-1} Y Y^{-1}\omega$$

$$Ek^* = \tfrac{1}{2}\omega^t Y^{-1}(\theta)\omega = \tfrac{1}{2}\omega^t I(\theta)\omega$$

At any configuration kinetic coenergy is a quadratic form in (generalized) velocity and its velocity gradient is the (generalized) momentum [8].

$$\mathbf{h} = I(\theta)\omega$$

For a generalized inertial system, Y is a symmetric, twice-contravariant tensor. To distinguish it from its inverse, the inertia tensor I, (symmetric, twice-covariant) Y will be termed the mobility tensor.

A knowledge of the geometric relation between coordinate frames is sufficient to transform any tensor from one frame to another. As the joint angles are a set of generalized coordinates, for any configuration of the linkage of Fig. 2 the end-point coordinates are related to the joint angles via the kinematic transformations.

$$\mathbf{X} = L(\theta)$$

Differentiating these transformations yields the relation between velocities (at any given configuration).

$$d\mathbf{X}/dt = \mathbf{V} = \mathbf{J}(\theta)\omega$$

$\mathbf{J}(\theta)$ in these equations is the configuration-dependent Jacobian. As the coordinate transformation does not store, dissipate or generate energy, incremental changes in energy are the same in all coordinate frames. This yields the relation between forces in each coordinate frame.

$$dEp = \mathbf{T}^t d\theta = \mathbf{F}^t d\mathbf{X} = \mathbf{F}^t\mathbf{J}(\theta)d\theta$$

At any given configuration

$$\mathbf{T} = \mathbf{J}^t(\theta)F$$

The same approach yields the relation between the momenta in each coordinate frame.

$$dEk = d\mathbf{h}^t\omega = d\mathbf{p}^t\mathbf{V} = d\mathbf{p}J(\theta)\omega$$

At any given configuration

$$\mathbf{h} = \mathbf{J}^t(\theta)\mathbf{p}$$

These relations may be used to express the mobility in end-point coordinates.

$$\mathbf{V} = \mathbf{J}\omega = \mathbf{J}\,Y\mathbf{J}^t\mathbf{p}$$

Denoting the end-point mobility by $W(\theta)$

$$W(\theta) = \mathbf{J}\,Y\mathbf{J}^t$$

$$\mathbf{V} = W(\theta)\mathbf{p}$$

The physical meaning of the mobility tensor is that if the system is at rest an applied force will produce an acceleration equal to the force vector premultiplied by the mobility tensor. At rest, $d\theta/dt = 0$ and hence:

$$d\mathbf{V}/dt = \mathbf{J}\,d\omega/dt$$

$$d\omega/dt = Y d\mathbf{h}/dt$$

From the generalized Hamiltonian [30]:

$$d\mathbf{h}/dt = \mathbf{T} - \nabla_\theta H$$

At rest, $\mathbf{h} = 0$ hence $H(\mathbf{h},\theta) = Ek = 0$ and $\nabla_\theta H = 0$. Thus:

$$d\mathbf{h}/dt = \mathbf{T}$$

$$d\mathbf{V}/dt = \mathbf{J}\,Y\mathbf{J}^t\mathbf{F} = W\,\mathbf{F}$$

As the mobility tensor is symmetric it may be diagonalized by rotating the coordinate axes to coincide with its eigenvectors. A force applied in the direction of an eigenvector (when the system is at rest) results in an acceleration in the same direction equal to the applied force multiplied by the corresponding eigenvalue. The eigenvalues represent the inverse of the apparent mass or inertia seen by the applied force or torque.

Because the kinetic energy is a quadratic form in momentum, it may be represented graphically by an ellipsoid (see Fig. 2), the ellipsoid of gyration [33]. This may be thought of as the set of all momenta which produce the same kinetic energy (an isokinetic contour in momentum space). The lengths of the principal axes of the ellipsoid of gyration are inversely proportional to the square roots of the eigenvalues, proportional to the square roots of the associated apparent mass or inertia. The long direction of the ellipsoid of Fig. 2 is the direction of the greatest apparent inertia.

In the general case when the system is not at rest the relation between applied force and resulting motion is (in general) nonlinear and must be written in terms of a complete set of state equations for the inertial system. A convenient set of state variables are the Hamiltonian states, generalized position (e.g., θ) and generalized momentum ($\mathbf{h}$). The state and output equations are in the form of generalized admittance (see Part I) as follows.

State equations:

$$d\mathbf{h}/dt = -\nabla_\theta[\tfrac{1}{2}\mathbf{h}^t\, Y(\theta)\,\mathbf{h}] + \mathbf{J}^t(\theta)\mathbf{F}$$

$$d\theta/dt = \nabla_{\mathbf{h}}[\tfrac{1}{2}h^t\, Y(\theta)\mathbf{h}] = Y(\theta)\mathbf{h}$$

Output equations (position and velocity):

$$X = L(\theta)$$

$$V = \mathbf{J}(\theta)\,Y(\theta)\mathbf{h}$$

APPENDIX III

Effect of Actuator Redundancy on Range of Feasible Stiffness

The differential stiffness tensor in relative joint angle coordinates $\{\rho_1, \rho_2\}$ due to the combined stiffnesses of monoarticular actuators, Ks, Ke and biarticular actuators Kt, is:

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} Ks+Kt & Kt \\ Kt & Ke+Kt \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}$$

The transformation from relative joint angle coordinates $\{\rho_1,\rho_2\}$ to absolute joint angle coordinates $\{\theta_1,\theta_2\}$ is:

$$\begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

Hence the stiffness tensor in absolute joint angle coordinates is:

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Ks+Kt & Kt \\ Kt & Ke+Kt \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} Ks+Ke & -Ke \\ -Ke & Kt+Ke \end{bmatrix}$$

The differential transformation from absolute joint angle coordinates $\{\theta_1,\theta_2\}$ to Cartesian end-point coordinates $\{X_1,X_2\}$ is:

$$\begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 & -L_2 \sin\theta_2 \\ L_1 \cos\theta_1 & L_2 \cos\theta_2 \end{bmatrix} \begin{bmatrix} d\theta_1 \\ d\theta_2 \end{bmatrix}$$

$$d\mathbf{X} = \mathbf{J}\, d\theta$$

To achieve an isotropic end-point stiffness (for which the corresponding potential function will have circular isopotentials) its eigenvalues must be equal. For simplicity assume each eigenvalue is unity.

$$Kx = 1$$

The corresponding stiffness tensor in absolute angle coordinates is:

$$K\mathrm{o} = \mathbf{J}^t Kx \mathbf{J} = \mathbf{J}^t \mathbf{J}$$

$$K\mathrm{o} = \begin{bmatrix} L_1{}^2 & L_1 L_2 \cos(\theta_2-\theta_1) \\ L_1 L_2 \cos(\theta_2-\theta_1) & L_2{}^2 \end{bmatrix}$$

To achieve an isotropic end-point stiffness it is necessary for the actual joint coordinate stiffness to equal the desired joint coordinate stiffness. Assuming $L_1 = L_2 = 1$ it can be seen that in the absence of biarticular actuators, i.e., $Kt = 0$, this condition is not satisfied except at:

$$\theta_2 - \theta_1 = 180°$$

point p in figure 3b. In contrast, given bi-articular actuators, i.e., $Kt \neq 0$, isotropic stiffness can be achieved throughout the region R in Fig. 3(c) defined by:

$$90° < \theta_2 - \theta_1 < 180°$$